

IB Questions Solution (Differential Equations)

①

#1. $x \frac{dy}{dx} - y^2 = 1$

$$x \frac{dy}{dx} = y^2 + 1$$

$$\frac{x}{dx} = \frac{y^2 + 1}{dy}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y^2 + 1}$$

$$\ln x + c = \arctan y. \quad (y=0, x=2)$$

$$\ln 2 + c = \arctan 0$$

$$c = -\ln 2.$$

$$\Rightarrow \arctan y = \ln x - \ln 2.$$

$$\arctan y = \ln \frac{x}{2}$$

$$y = \tan\left(\ln \frac{x}{2}\right)$$

#2. $x(s) = \arctan(\sin s)$

$$a = \frac{dx}{dt} = \frac{dx}{ds} \cdot \frac{ds}{dt}$$

$$(a) \quad = \frac{dx}{ds} \cdot v$$

$$= \frac{d \arctan(\sin s)}{ds} \cdot v$$

$$= \frac{1}{1 + (\sin s)^2} \cdot (\cos(s) \cdot \arctan(\sin s))$$

$$a(s) = \frac{\cos(s) \cdot \arctan(\sin(s))}{1 + (\sin(s))^2}$$

$$(b) \quad 0.25 = \frac{\cos(s) \cdot \arctan[\sin s]}{1 + \sin^2 s}$$

Solve by nsolve.
GDC.

$$s \approx -2.97$$

#3. $a = \frac{2s}{s^2 + 1}$

$$a(t) = \frac{dx}{dt} = \frac{dx}{ds} \cdot \frac{ds}{dt} = \frac{dx}{ds} \cdot v$$

(a)

$$\Rightarrow \frac{2s}{s^2 + 1} = \frac{dx}{ds} \cdot v \quad \rightarrow \quad \ln|s^2 + 1| = \frac{1}{2} x^2 + C$$

$$\Rightarrow \int \frac{2s}{s^2 + 1} \cdot ds = \int x \cdot du$$

($u = s^2 + 1 \quad du = 2s \cdot ds$)

$$\ln 2 = \frac{1}{2} \cdot 2^2 + C$$

$$C = \ln 2 - 2.$$

$$s=1 \quad \downarrow \quad v=2.$$

→

$$\frac{1}{2}v^2 = \ln(S^2 + 1) + 2 - \ln 2.$$

$$\frac{1}{2}v^2 = \ln\left(\frac{S^2 + 1}{2}\right) + 2.$$

$$v^2 = 2 \ln\left(\frac{S^2 + 1}{2}\right) + 4$$

$$v = \pm \sqrt{2 \ln\left(\frac{S^2 + 1}{2}\right) + 4} \quad v \geq 0.$$

$$(b) \quad v = \sqrt{2 \ln\left(\frac{86}{2}\right) + 4}$$

$$v = \sqrt{2 \ln 13 + 4} \quad \frac{m}{s}$$

#4.

$$(a) \quad (i) \quad v(t) = -10t$$

$$(ii) \quad v(10) = -100 \frac{m}{s}.$$

$$(iii) \quad S(100) \Rightarrow S(t) = \int -10t \, dt = -5t^2 + C. \quad C = 1000$$

$$S(t) = -5t^2 + 1000$$

$$S(10) = -5(10)^2 + 1000 = 500$$

(b) $a(x) = -10 - 5v \quad t \geq 10$

$$\Rightarrow \frac{dx}{dv} = \frac{1}{\frac{dv}{dx}} = \frac{1}{a(x)} = \frac{1}{-10-5v}$$

$$\int dx = \int \left(\frac{1}{-10-5v} \right) dv$$

$$x = -\frac{1}{5} \ln | -10-5v | + C \quad \Leftarrow \begin{matrix} v(10) = -100 \\ (x=10, v=-100) \end{matrix}$$

$$10 = -\frac{1}{5} \ln | -10-5(-100) | + C$$

$$C = 10 + \frac{1}{5} \ln(490)$$

(c) $x = \frac{1}{5} \ln(490) - \frac{1}{5} \ln | -10-5v | + 10$

$$= \frac{1}{5} \ln \left[\frac{490}{-10-5v} \right] + 10$$

$$\boxed{x = 10 + \frac{1}{5} \ln \left[\frac{98}{-2-v} \right]}$$

(d) $\frac{1}{5} \ln \left[\frac{98}{-2-v} \right] = x - 10$

$$\Rightarrow \ln \left[\frac{98}{-2-v} \right] = 5(x-10)$$

$$\Rightarrow \frac{98}{-2-v} = e^{5(x-10)}$$

$$\Rightarrow \frac{-98}{e^{5(x-10)}} = 2+v \Rightarrow \boxed{v = \frac{-98}{e^{5(x-10)}} - 2} \quad \text{OR} \quad \boxed{v = -98 e^{-5(x-10)} - 2}$$

$$(e) \quad S = \int v \, dt$$

$$= \int (-98 e^{5(10-t)} - 2) \, dt$$

$$= \frac{-98}{-5} e^{5(10-t)} - 2t + C \quad \Leftarrow \quad \left(\begin{array}{l} t=10 \\ S=500 \end{array} \right) \text{ from (a)}$$

$$\hookrightarrow 500 = \frac{98}{5} \cdot e^0 - (2)(10) + C$$

$$C = 500 + 20 - \frac{98}{5}$$

$$C = 500.4$$

$$S(t) = \frac{98}{5} e^{5(10-t)} - 2t + 500.4$$

$$(f) \quad 0 = \frac{98}{5} e^{5(10-t)} - 2t + 500.4 \quad \Leftarrow \text{GDC}$$

n-solve. $t = ?$ use n-solve.