Implicit Differentiation Notes

When an equation is not a function, it is often difficult or impossible to write as y=. Such relationships between x and y are called implicit relations.

Examples:

Explicit Equation:

$$y = \left(6x^5 + 4x^2\right)^3$$

Implicit relations (or implicit equations):

$$x^2 + y^2 = 25$$

$$x\sqrt{y} - y^4 = 2x$$

The method of finding $\frac{dy}{dx}$ of implicit equations is "Implicit Differentiation".

Practice of the chain rule notation:

1. $\frac{d}{dx}\left(u^5\right)$

$= \frac{du^5}{du} \cdot \frac{dy}{dx}$

$= 5u^4 \cdot \frac{dy}{dx}$

2. $\frac{d}{dx}\left(y^3\right)$

$= \frac{dy^3}{dy} \cdot \frac{dy}{dx}$

$= 3y^2 \cdot \frac{dy}{dx}$

$$\therefore \frac{d}{dx}\left(y^n\right) = n(y^{n-1})(\frac{dy}{dx})$$

Implicit Differentiation Examples)

Find $\frac{dy}{dx}$ if…

3. $x^2 + y^2 = 25$

$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$

$2x + \frac{d}{dy}(y^2) \cdot \frac{dy}{dx} = 0$

$2x + 2y \cdot \boxed{\frac{dy}{dx}} = 0$

$2y \cdot \frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y} = \boxed{\frac{-x}{y}}$

4. $x\sqrt{y} - y^4 = 2x$

$\frac{d}{dx}[x \cdot \sqrt{y}] - \frac{d}{dx}(y^4) = \frac{d}{dx}(2x)$

$(\frac{d}{dx}x) \cdot \sqrt{y} + x \cdot \frac{d}{dx}(y)^{\frac{1}{2}} - 4 \cdot y^3 \cdot \frac{dy}{dx} = 2$

$1 \cdot \sqrt{y} + x(\frac{1}{2})(y)^{\frac{-1}{2}} \cdot \boxed{\frac{dy}{dx}} - 4y^3 \boxed{\frac{dy}{dx}} = 2.$

$\cancel{\sqrt{y}} + \frac{x}{2\sqrt{y}} \cdot \frac{dy}{dx} - 4y^3 \cdot \frac{dy}{dx} = 2 - \sqrt{y}$

$-\sqrt{y}$

$\frac{dy}{dx}\left[\frac{x}{2\sqrt{y}} - 4y^3\right] = 2 - \sqrt{y}$

$\frac{dy}{dx} = \frac{(2-\sqrt{y}) \cdot 2\sqrt{y}}{\left(\frac{x}{2\sqrt{y}} - 4y^3\right) 2\sqrt{y}}$

$\frac{dy}{dx} = \boxed{\frac{4\sqrt{y} - 2y}{x - 8y^3\sqrt{y}}}$

5. Find the slope of the tangent to $y^3 + x^2 = 5$ at $(-2,1)$.

$y^3 + x^2 = 5$

$y' = \frac{dy}{dx}$

$3y^2 \cdot \boxed{y'} + 2x = 0$

$3y^2 y' = -2x$

$y' = \boxed{\frac{-2x}{3y^2}}$

$y'(-2,1) = \frac{(-2)(-2)}{3 \cdot 1^2} = \boxed{\frac{4}{3}}$

More Examples)

6. $\frac{d}{dx}(xy^2) = \frac{d}{dx}(x) \cdot y^2 + x \cdot \frac{d}{dx} y^2$ $\rightarrow \frac{d}{dy}(y^2) \cdot \frac{dy}{dx}$

$= \boxed{y^2 + x \cdot 2y \cdot \frac{dy}{dx}}.$

7. Find $\frac{dy}{dx}$ if $x + x^2 y + y^3 = 100$

$\frac{d}{dx} x + \frac{d}{dx}(x^2 y) + \frac{d}{dx} y^3 = \frac{d}{dx} 100.$

$1 + 2x \cdot y + x^2 \boxed{\frac{dy}{dx}} + 3y^2 \cdot \boxed{\frac{dy}{dx}} = 0$

$\frac{dy}{dx}[x^2 + 3y^2] = -1 - 2xy$

$\boxed{\frac{dy}{dx} = \frac{-1-2xy}{(x^2+3y^2)}}$

8. 7. Find $\frac{dy}{dx}$ if 2. $x^2 - 2xy + y^2 = x$

$x^2 - 2xy + y^2 = x$

$2x - 2[y + x \cdot \frac{dy}{dx}] + 2y \frac{dy}{dx} = 1$

$\boxed{\frac{dy}{dx} = \frac{1+2y-2x}{2y-2x}}$