3. Determine what makes the integrals below "improper". Then, determine how to rewrite the integrals using "proper" limit notation. Finally, find the area by evaluating the limit and the integral.

(a) $\int_1^\infty \frac{1}{x^2}\,dx$

$$\Rightarrow \lim_{a\to\infty}\int_1^a \frac{1}{x^2}\,dx = \lim_{a\to\infty}\left[\frac{-1}{x}\right]_1^a$$

$$= \lim_{x\to a}\left[\frac{-1}{a}+1\right] = 1 \quad \text{(converge)}$$

(b) $\int_2^6 \frac{1}{\sqrt{x-2}}\,dx$

$$\Rightarrow \lim_{a\to 2^+}\int_a^6 \frac{1}{\sqrt{x-2}}\,dx = \lim_{a\to 2^+}\left[2\sqrt{x-2}\right]_a^6$$

$$= \lim_{a\to 2^+}\left[2\sqrt{a-2}-4\right] = \boxed{4.} \quad \text{(converge)}$$

(c) $\int_{-\infty}^0 e^x\,dx$

$$\Rightarrow \lim_{a\to-\infty}\int_a^0 e^x\,dx = \lim_{a\to-\infty}\left[e^0 - e^a\right]$$

$$= 1 - \frac{1}{e^\infty} = \boxed{1}$$

(converge)

(d) $\int_0^2 \frac{1}{x^2-2x+1}\,dx$ *(hint: Break into two integrals)*

$$\Rightarrow \lim_{a\to 1^-}\int_0^a \frac{1}{(x-1)^2}\,dx + \lim_{a\to 1^+}\int_a^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{a\to 1^+}\left[\frac{-1}{x-1}\right]_0^a + \lim_{a\to 1^+}\left[\frac{-1}{x-1}\right]_a^2$$

$$\lim_{a\to 1^-}\left[\frac{-1}{a-1}+1\right] + \lim_{a\to 1^+}\left[-1+\frac{1}{a-1}\right] = \infty \quad \text{(diverge)}$$

Part III: Additional Practice:

(a) $\int_0^\infty \frac{1}{x^2+1}\,dx$

$$\Rightarrow \lim_{a\to\infty}\int_0^a \frac{1}{x^2+1}\,dx$$

$$= \lim_{a\to\infty}\left[\tan^{-1}x\right]_0^a$$

$$= \lim_{a\to\infty}\left[\tan^{-1}a - \tan^{-1}0\right] = \frac{\pi}{2} \quad \text{(converge)}$$

(b) $\int_1^\infty \frac{x}{e^x}\,dx$

$$\Rightarrow \lim_{a\to\infty}\int_1^a \frac{x}{e^x}\,dx$$

u	dv
x (+)	e^{-x}
1 (−)	$-e^{-x}$
0	e^{-x}

$$= \lim_{a\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^a$$

$$= \lim_{a\to\infty}\left[\frac{-x}{e^x} - \frac{1}{e^x}\right]_1^a = \lim_{a\to\infty}\left[\frac{-a}{e^a} - \frac{1}{e^a}\right] - \left[\frac{-1}{e} - \frac{1}{e}\right] = \frac{2}{e}$$

(c) $\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$ *(Challenging)*:

$$\Rightarrow \lim_{a\to-\infty}\int_a^0 \frac{e^x}{1+(e^x)^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{e^x}{1+(e^x)^2}\,dx.$$

$$= \lim_{a\to-\infty}\tan^{-1}(e^x)\Big|_a^0 + \lim_{b\to\infty}\tan^{-1}(e^x)\Big|_0^b$$

$$\to \tan^{-1}(0) - (\tan^{-1}(e^{-\infty})) + \tan^{-1}(\infty) - \tan^{-1}(0) = \boxed{\frac{\pi}{2}} \quad \text{(converges)}$$