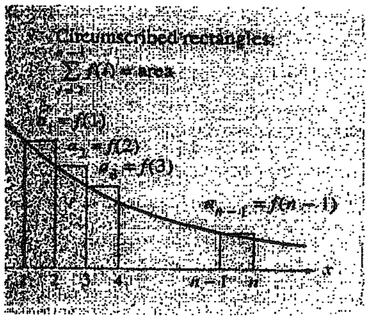
IB Math 3: Integral Test Name: *Key* Period: ______ ①

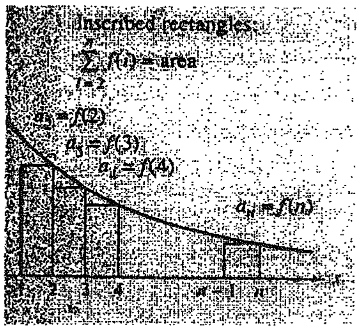
Part I: Consider $\sum_{k=1}^{\infty} 5^{-x}$ and $\int_1^{\infty} 5^{-x}dx$.

1. Set up the area under the curve, $f(x)=5^{-x}$, in the interval $[1, n-1]$ using sigma notation and left rectangle as shown.



$\sum_{i=1}^{n-1} 5^{-i}$ OR $\sum_{x=1}^{n-1} 5^{-x}$

2. Set up the area under the curve, $f(x)=5^{-x}$, in the interval $[2, n]$ using sigma notation and right rectangle as shown.



$\sum_{i=2}^{n} 5^{-i}$ $\sum_{x=2}^{n} 5^{-x}$

3. Express the exact area under the curve, $f(x)=5^{-x}$, in the interval $[1, n]$.

4. Compare the areas of circumscribed rectangles from (1), inscribed rectangles from (2), and the exact area from (3) by filling up the boxes.

$$\boxed{\sum_{x=2}^{n} 5^{-x}} \le \boxed{\int_1^n 5^{-x}dx} \le \boxed{\sum_{x=1}^{n-1} 5^{-x}}$$

5. Assume the infinite sum, $n \to \infty$, of the area from (1) converges to S. What can you conclude for the infinite sum of the area from (2)? Consequently, what can you conclude for the area of $\int_1^{\infty} f(x)dx$?

(2) $\lim_{n\to\infty} \sum_{x=2}^{n} 5^{-x} = S$

$$\therefore \lim_{n\to\infty} \sum_{x=2}^{n} 5^{-x} \le \int_1^{\infty} 5^{-x}dx \le \lim_{n\to\infty} \sum_{x=1}^{n-1} 5^{-x}$$

$$\Rightarrow S \le \int_1^{\infty} 5^{-x}dx \le S \Rightarrow \int_1^{\infty} 5^{-x}dx \text{ converges to } S.$$

IB Math 3: Summary Notes: Integral Test

If $a_n = f(n)$ for $n = 1, \quad 2, \quad 3, \quad 4.......$
where f(x) is a positive, continuous, and decreasing function of x for $x \ge N$, for some number N,

then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)dx$ either both converge or both diverge.

- $\lim_{n\to\infty} \int_1^n f(x)dx = \infty \Rightarrow \lim_{n\to\infty} S_n = \infty$ where $S_n = \sum_{n=1}^{n} f(n)$: Diverges
- $\lim_{n\to\infty} \int_1^n f(x)dx = L \Rightarrow \lim_{n\to\infty} S_n = L$ where $S_n = \sum_{n=1}^{n} f(n)$: Converges

Example) Test the series $\sum_{k=1}^{\infty} \frac{1}{k}$ for convergence.

Notes: $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} .... + \frac{1}{\infty}$
is known as Harmonic Series (Vibrating String)

Solution:

1) $a_k = \frac{1}{k} > 0$ for $k \ge 1$ positive

$a'(k) = \frac{-1}{k^2} < 0$ for $k \ge 1 \Rightarrow$ decreasing.

2) Integral test : $a_k \Rightarrow \frac{1}{x}$ for $x \ge 1$

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{a\to\infty} \int_1^a \frac{1}{x} dx = \lim_{a\to\infty} [\ln x]_1^a$$

$$= \lim_{a\to\infty} [\ln a - 0] = \infty$$

$\therefore \sum^{\infty} \frac{1}{k}$ diverges

6. Assume the infinite sum, $n \to \infty$, of the area from (1) diverges to ∞. What can you conclude for the infinite sum of the area from (2)? Consequently, what can you conclude for the area of $\int_1^{\infty} f(x)dx$?

$$\lim_{n\to\infty} \sum_{x=1}^{n-1} 5^{-x} \le \int_1^n 5^{-x}dx \le \lim_{n\to\infty} \sum_{x=2}^{n} 5^{-x}$$

$$\Rightarrow \quad \infty \le \int_1^n 5^{-x}dx \le \infty \quad \Rightarrow \therefore \int_1^n 5^{-x}dx \text{ diverges to } \infty.$$

Part II: Consider P-Series: $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} \ldots\ldots\ldots$

1. Evaluate $\lim_{a\to\infty} \int_1^a \frac{1}{x^p} dx \Rightarrow \lim_{a\to\infty} \left[\frac{1}{1-p} x^{1-p}\right]_1^a = \lim_{a\to\infty} \left[\frac{1}{1-p} a^{1-p} - \frac{1}{1-p}\right]$

$$= \lim_{a\to\infty} \left[\frac{1}{1-p}\left(a^{1-p}-1\right)\right] \Rightarrow \begin{cases} \text{If } p=1 \Rightarrow \text{diverges} \\ \text{If } p>1 \Rightarrow \text{converges} \\ \text{If } p<1 \Rightarrow \text{diverges.} \end{cases}$$

$\therefore$ Examples are also okay such as $p=1$, $p=5$, $p=-5$.

2. For what values of p, the above limit will converse? And For what values of p, the above limit will diverge?

when $p=1 \Rightarrow \lim_{a\to\infty} \int_1^a \frac{1}{x}dx = \lim_{a\to\infty}\left[\ln x\right]_1^a = \lim_{a\to\infty}\left[\ln a\right] = \infty$ diverges.

$p>1 \Rightarrow \lim_{a\to\infty}\left[\frac{1}{1-p} a^{1-p}-1\right] = 0$ converges

$p<1 \Rightarrow \lim_{a\to\infty}\left[\frac{1}{1-p} a^{1-p}-1\right] = \infty$ diverges.

3. Evaluate the followings and confirm if your answers from (2) are correct. And discuss the results.

a. $\lim_{a\to\infty} \int_1^a \frac{1}{x^2} dx$

$$= \lim_{a\to\infty}\left[\frac{-1}{x}\right]_1^a = \lim_{a\to\infty}\left[\frac{-1}{a}+1\right] = \frac{-1}{\infty} = 0.$$

converges.

b. $\lim_{a\to\infty} \int_1^a \frac{1}{\sqrt{x}} dx$

$$= \lim_{a\to\infty}\left[2x^{\frac{1}{2}}\right]_1^a$$

$$= \lim_{a\to\infty}\left[2a^{\frac{1}{2}}-2\right] = \infty$$

diverges.

Answer key)

③

c. $\lim_{a\to\infty}\int_1^a \frac{1}{x}\,dx$

$= \lim_{a\to\infty}\left[\ln x\right]_1^a$

$= \lim_{a\to\infty}\left[\ln a\right] = \infty$

diverges

d. $\lim_{a\to\infty}\int_1^a \frac{1}{\sqrt[4]{x}}\,dx$

$= \lim_{a\to\infty}\left[\frac{4}{3}x^{\frac{3}{4}}\right]_1^a$

$= \lim_{a\to\infty}\left[\frac{4}{3}a^{\frac{3}{4}} - \frac{4}{3}\right] = \infty$

diverges.

4. Consider convergence of the P-Series, $\sum_{k=1}^{\infty}\frac{1}{k^p}$. And then complete the p-series theorem.

The p-series converges if $p>1$
The p-series diverges if $p\le 1$

5. Describe the characteristics of P-series in general?

$f(x) = \frac{1}{x^p}$ ⇒ When $p>1$ ⇒ decreasing, monotonic, & positive.
when $p\le 1$ ⇒ Increasing, monotonic, & positive.

6. Determine if the following series converges or diverges by the Integral Test.

(a) $\sum_{k=1}^{\infty}\frac{1}{n^2+1}$ ⇒ converges

Integral test ⇒ $f(x)=\frac{1}{x^2+1}$ (decreasing & positive function)

$\lim_{a\to\infty}\int_1^a \frac{1}{x^2+1}\,dx$

$= \lim_{a\to\infty}\left[\tan^{-1}x\right]_1^a = \lim_{a\to\infty}\left[\tan^{-1}a - \tan^{-1}1\right] = \frac{\pi}{2}-\frac{\pi}{4} = \boxed{\frac{\pi}{4}}$

(b) $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k^3}}$ ⇒ converges.

Integral test ⇒ $f(x)=\frac{1}{\sqrt{x^3}}$ (decreasing & positive function)

$\lim_{a\to\infty}\int_1^a \frac{1}{x^{\frac{3}{2}}}\,dx = \lim_{a\to\infty}\left[-2x^{-\frac{1}{2}}\right]_1^a$

$= \lim_{a\to\infty}\left[\underbrace{\frac{-2}{\sqrt{a}}}_{0} + 2\right] = 2$

(c) $\sum_{k=1}^{\infty}\frac{k}{e^k}$ ⇒ converges.

Integral test ⇒ $f(x)=\frac{x}{e^x}$ (decreasing & positive)

$\lim_{a\to\infty}\int_1^a xe^{-x}\,dx = \lim_{a\to\infty}\left[-xe^{-x}-e^{-x}\right]_1^a$

$= \lim_{a\to\infty}\left[\underbrace{\frac{-a}{e^a}}_{0} - \underbrace{\frac{1}{e^a}}_{0}\right] - \left[\frac{-1}{e}-\frac{1}{e}\right] = \frac{2}{e}$