#1. $\sin^2 2x = 1 - \cos^2 2x.$

$\cos^2 2x = 1 - \sin^2 2x$

I.

$$\int (1-\cos^2 2x)\sin 2x \cdot \cos^5 2x \, dx.$$

$u = \cos 2x.$

$du = -2\sin 2x\, dx$

$-\frac{1}{2} du = \sin 2x\, dx$

$$-\frac{1}{2}\int (1-u^2)\, u^5\, du$$

$$= -\frac{1}{2}\int (u^5 - u^7)\, du$$

$$= -\frac{1}{2}\left[\frac{1}{6}u^6 - \frac{1}{8}u^8\right] + C$$

$$= \boxed{-\frac{1}{12}\cos^6 2x + \frac{1}{16}\cos^8 2x + C}$$

II

$$\int \sin^3 2x\,(1-\sin^2 2x)^2 \cos 2x\, dx$$

$u = \sin 2x$

$du = 2\cos 2x\, dx.$

$\frac{1}{2} du = \cos 2x\, dx$

$$\frac{1}{2}\int u^3 (1-u^2)^2\, du$$

#2. $\int (\tan^2 x)(\tan^2 x)\,dx$.

$= \int (\sec^2 x - 1)\tan^2 x\,dx$.

$= \int \sec^2 x \cdot \tan^2 x\,dx - \int \tan^2 x\,dx$.

$= \int \sec^2 x \tan^2 x\,dx - \int (\sec^2 x - 1)\,dx$.

$\left(\begin{array}{l} u = \tan x \\ du = \sec^2 x\,dx \end{array} \right)$

$= \int u^2\,du - \tan x + x + C$

$= \boxed{\frac{1}{3}\tan^3 x - \tan x + x + C}$

3. $\int \cos^2 x \sin^2 x \, dx$.

1) $\boxed{\cos 2x = 1 - 2\sin^2 x}$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\boxed{\cos 2x = 2\cos^2 x - 1}$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$

$\sin 2x = 2\sin x \cos x$.

$$= \int \frac{1}{2}(1+\cos 2x)\frac{1}{2}(1-\cos 2x)\,dx.$$

$$= \frac{1}{4}\int (1 - \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2}(1+\cos 4x)\right)dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right)dx.$$

$$= \boxed{\frac{1}{4}\left[\frac{1}{2}x - \frac{1}{8}\sin 4x\right] + C}$$ ✓

$$= \frac{1}{4}\left[\frac{1}{2}x - \cancel{\frac{1}{8}\cdot 2}\sin 2x \cos 2x\right] + C.$$

$$= \frac{1}{4}\left[\frac{1}{2}x - \frac{1}{4}2\sin x\cos x(1 - 2\sin^2 x)\right] + C.$$