IB Math 3: Exit slip

Name: key Period: _____

1. Limit applying L'Hopital's rule:

a. $\lim_{x\to 0}\dfrac{x+\sin(x^2+x)}{3x+\sin x}$

b. $\lim_{x\to 0^+}(e^x+x)^{1/x}$

c. $\lim_{x\to\infty}(1-\dfrac{2}{3x})^x$

2. mproper integrals: Evaluate if converges.

a. $\displaystyle\int_3^{+\infty}\frac{dx}{\sqrt[3]{2x-1}}$

b. $\displaystyle\int_0^{+\infty}x^2e^{-x}dx$

c. $\displaystyle\int_{-\infty}^{+\infty}\frac{dx}{x^2+1}$

#1

a) $$\lim_{x\to 0}\frac{x+\sin(x^2+x)}{3x+\sin x}\ \left(=\frac{0}{0}\right) \xrightarrow{\text{L'Hôpital's Rule}} \lim_{x\to 0}\frac{1+[\cos(x^2+x)](2x+1)}{3+\cos x}=\frac{1+1}{3+1}=\frac{2}{4}$$

$$=\boxed{\frac{1}{2}}$$

b. $$\lim_{x\to 0^+}(e^x+x)^{\frac{1}{x}}=L$$

$$\Rightarrow \lim_{x\to 0^+}\ln(e^x+x)^{\frac{1}{x}}=\ln L$$

$$\Rightarrow \lim_{x\to 0^+}\frac{\ln(e^x+x)}{x}\ \left(=\frac{0}{0}\right)=\ln L$$

L'Hopital's Rule

$$\Rightarrow \lim_{x\to 0^+}\frac{e^x+1}{1}=\ln L$$

$$\Rightarrow 2=\ln L \Rightarrow \boxed{L=e^2}$$

c. $$\lim_{x\to\infty}\left(1-\frac{2}{3x}\right)^x=L$$

$$\Rightarrow \lim_{x\to\infty}x\ln\left(1-\frac{2}{3x}\right)=\ln L$$

$$\Rightarrow \lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{3x}\right)}{\frac{1}{x}}\ \left(=\frac{0}{0}\right)=\ln L$$

L'Hopital's Rule

$$\Rightarrow \lim_{x\to\infty}\frac{\left(\frac{1}{1-\frac{2}{3x}}\right)\left(\frac{-2}{3}\cdot\cancel{\frac{-1}{x^2}}\right)}{\cancel{\frac{-1}{x^2}}}=\ln L$$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{1}{1-\frac{2}{3x}\to 0}\right)\left(\frac{-2}{3}\right)=\ln L$$

$$\Rightarrow \frac{-2}{3}=\ln L \Rightarrow L=e^{-\frac{2}{3}}$$

Exit slip.

2.

a. $\displaystyle\int_3^\infty \frac{dx}{\sqrt[3]{2x-1}}$

$$= \lim_{a\to\infty} \int_3^a (2x-1)^{-\frac{1}{3}}\,dx$$

$$= \lim_{a\to\infty} \left[\frac{3}{2}\cdot\frac{1}{2}\,(2x-1)^{\frac{2}{3}}\right]_3^a$$

$$= \lim_{a\to\infty} \left[\frac{3}{4}\,[2a-1]^{\frac{3}{2}} - \frac{3}{4}\,(5)^{\frac{2}{3}}\right]$$

$= \boxed{\infty}$ (Diverges)

b. $\displaystyle\int_0^\infty x^2 e^{-x}\,dx$

u	dv
x^2	e^{-x}
$2x$	$-e^{-x}$
2	e^{-x}
0	$-e^{-x}$

$+$ $-$ $+$

$$= \lim_{a\to\infty} \left[\int_0^a x^2 e^{-x}\,dx\right]$$

$$= \lim_{a\to\infty} \left[-\frac{x^2}{e^x} - \frac{2x}{e^x} - \frac{2}{e^x}\right]_0^a$$

$$= \lim_{a\to\infty} \left[\underbrace{-\frac{a^2}{e^a}}_{\text{L'Hopital}}{}^{\frac{\infty}{\infty}} - \underbrace{\frac{2a}{e^a}}_{\text{L'Hopital}}{}^{\frac{\infty}{\infty}} - \cancel{\frac{2}{e^x}}^{\,0} + 2\right]$$

$$= \lim_{a\to\infty} \left[\underbrace{\frac{-2a}{e^a}}_{\text{L'Hopital}}{}^{\frac{\infty}{\infty}} - \cancel{\frac{2}{e^a}}\right]^{0} + 2$$

$$= \lim_{a\to\infty} \left[\cancel{\frac{-2}{e^a}}\right]^{0} + 2 \quad = \boxed{2} \quad \text{(Converges)}$$

c. $$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{0} \frac{dx}{x^2+1} + \int_0^\infty \frac{dx}{x^2+1}$$

$$= \lim_{a\to-\infty} \Big[\arctan x\Big]_a^0 + \lim_{b\to\infty} \Big[\arctan x\Big]_0^b$$

$$= \lim_{a\to-\infty} \left[\arctan \cancel{b}\,0 - \arctan a\right] + \lim_{b\to\infty} \left[\arctan b - \arctan 0\right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

#5

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$

$$= \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{e^x}{1+e^{2x}}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{e^x}{1+e^{2x}}\,dx \Leftarrow \left(\begin{array}{l} u = e^x \\ du = e^x dx \\ u^2 = e^{2x} \end{array} \right.$$

$$= \lim_{a\to-\infty} \Big[\arctan e^x \Big]_a^0 + \lim_{b\to\infty} \Big[\arctan e^x \Big]_0^b$$

$$= \lim_{a\to-\infty} \Big[\underbrace{\arctan e^0}_{\frac{\pi}{4}} - \underbrace{\arctan e^a}_{\arctan e^{-\infty} \to 0} \Big] + \lim_{b\to\infty} \Big[\underbrace{\arctan e^b}_{\arctan e^{\infty} \to \frac{\pi}{2}} - \arctan e^0 \Big]$$

$$= \cancel{\frac{\pi}{4}} - 0 + \frac{\pi}{2} - \cancel{\frac{\pi}{4}} = \boxed{\frac{\pi}{2}}$$

#5.

☆ (b) $\int_1^{\infty} \frac{x}{e^x}\,dx$.

$$\lim_{a\to\infty} \int_1^a \frac{x}{e^x}\,dx.$$

u	du
x	e^{-x} (+)
1	$-e^{-x}$ (−)
0	e^{-x}

$$= \lim_{a\to\infty} \left[-x \cdot e^{-x} - e^{-x} \right]_1^a$$

$$= \lim_{a\to\infty} \left[\underbrace{\frac{-a}{e^a}}_{=\left(\frac{\infty}{\infty}\right)} - \cancel{\frac{1}{e^a}}^{\,0} + \frac{1}{e} + \frac{1}{e} \right]$$

$$= \lim_{a\to\infty} \left[\cancel{\frac{-1}{e^a}}^{\,0} \right] + \frac{2}{e} = \frac{2}{e}$$