l'Hôpital's rule:

Theorem: Let $f(x)$ and $g(x)$ be the functions that are differentiable on an open interval (a,b) containing c. Assum that $g'(x) \neq 0$ for all x in (a,b). If the limit of $\dfrac{f(x)}{g(x)}$ as x approaches c produces the indeterminate form $\lim \dfrac{0}{0}$, $\lim \dfrac{\infty}{\infty}$, and $\lim 0 \cdot \infty$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

Proof: Suppose $f(x)$ and $g(x)$ are differentiable function for $x \in R$.
and $\underline{f(c) = g(c) = 0}$. $\left(\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{0}{0} \right)$

$f(x) \approx L(x) = f'(c)(x-c) + f(c)$

$g(x) \approx G(x) = g'(c)(x-c) + g(c)$

$$\Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{L(x)}{G(x)} = \lim_{x\to c} \frac{f'(c)(x-c) + \overset{0}{f(c)}}{g'(c)(x-c) + \underset{0}{g(c)}} = \lim_{x\to c} \frac{f'(c)\cancel{(x-c)}}{g'(c)\cancel{(x-c)}}$$

$$\therefore \boxed{\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}} = \frac{f'(c)}{g'(c)}$$

Example 1) $\lim_{x\to 0} \dfrac{\sin x}{x}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{\cos x}{1} = \frac{1}{1} = \boxed{1}$$

Example 2) $\lim_{x\to\infty} \dfrac{2x^2 - 3x + 1}{3x^2 + 5x - 2}$ $\left(\frac{\infty}{\infty}\right)$

$$= \lim_{x\to\infty} \frac{4x-3}{6x+5} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{4}{6} = \boxed{\frac{2}{3}}$$

Example 3) $\lim_{x\to 0} \dfrac{(1-\cos x)\sin 4x}{x^3 \cos x}$

$$= \lim_{x\to 0} \frac{(1-\cos x)}{x^2} \left(\frac{0}{0}\right) \lim_{x\to 0} \frac{\sin 4x}{x} \left(\frac{0}{0}\right) \cdot \lim_{x\to 0} \frac{1}{\cos x} \to 1$$

$$= \lim_{x\to 0} \frac{\sin x}{2x} \left(\frac{0}{0}\right) \lim_{x\to 0} \frac{4\cos 4x}{1} \cdot 1$$

$$= \lim_{x\to 0} \frac{\cos x}{2} \cdot 4 \cdot 1 = \frac{1}{2} \cdot 4 \cdot 1 = \boxed{2}$$

Example 4) $\lim_{x\to\infty} x \sin \dfrac{1}{x}$

$\left(\infty \cdot \sin \frac{1}{\infty}\right) \Rightarrow (\infty \cdot 0)$

$$\left(u = \frac{1}{x} \Rightarrow x = \frac{1}{u} \atop x \to \infty \Rightarrow u \to 0 \right) \Rightarrow \lim_{u\to 0} \frac{1}{u} \sin u = \lim_{u\to 0} \frac{\sin u}{u} \left(\frac{0}{0}\right) = \lim_{u\to 0} \frac{\cos u}{1}$$

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$$\Rightarrow \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{L(x)}{G(x)} = \lim_{x \to c} \frac{f'(c)(x-c) + \cancel{f(c)}^{\,0}}{g'(c)(x-c) + \cancel{g(c)}^{\,0}} = \lim_{x \to c} \frac{f'(c)\cancel{(x-c)}}{g'(c)\cancel{(x-c)}}$$

$$\therefore \boxed{\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}} = \frac{f'(c)}{g'(c)}$$

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Example 3) $\lim\limits_{x \to 0} \dfrac{(1-\cos x)\sin 4x}{x^3 \cos x}$ $= \boxed{\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2}} \left(\dfrac{0}{0}\right) \lim\limits_{x \to 0} \dfrac{\sin 4x}{x} \left(\dfrac{0}{0}\right) \cdot \lim\limits_{x \to 0} \dfrac{1}{\cos x} \to 1$

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$\left(\infty \cdot \sin \frac{1}{\infty}\right) \Rightarrow (\infty \cdot 0)$

$\left(\begin{array}{l} y = \frac{1}{x} \Rightarrow x = \frac{1}{y} \\ x \to \infty \Rightarrow y \to 0 \end{array} \right) \Rightarrow \lim\limits_{y \to 0} \dfrac{1}{y} \sin y = \lim\limits_{y \to 0} \dfrac{\sin y}{y} \left(\dfrac{0}{0}\right) = \lim\limits_{y \to 0} \dfrac{\cos y}{1}$ $= \boxed{1}$

Example 5) $\ln \lim_{x\to+\infty}(1+\frac{1}{x})^x = \ln L$

$\lim_{x\to\infty} \ln(1+\frac{1}{x})^{x} = \ln L$

$= \lim_{x\to\infty} x \ln(1+\frac{1}{x}) = \ln L$

$= \lim_{x\to\infty} \frac{\ln(1+\boxed{\frac{1}{x}})}{\boxed{\frac{1}{x}}} = \lim \frac{\ln(1+\frac{1}{\infty})}{\frac{1}{\infty}} = \lim \frac{0}{0}$

$= \lim_{x\to\infty} \frac{(\ln(1+\frac{1}{x}))'}{(\frac{1}{x})'} = \lim_{x\to\infty} \frac{(\frac{1}{1+\frac{1}{x}})(\cancel{\frac{-1}{x^2}})}{\cancel{\frac{-1}{x^2}}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1+\frac{1}{\infty}} = \boxed{1}$

$1 = \ln L$

$\boxed{L = e}$

Use l'Hopital's rule to find each limit. Show your work clearly and circle your final answer.

1. $\lim_{x\to 1}\frac{x^3-1}{x^2-1}$

2. $\lim_{x\to 3}\frac{x^3-27}{x^2-9}$

3 $\lim_{x\to 1}\frac{x^{10}-1}{x-1}$

4. $\lim_{x\to 0}\frac{1-\cos^2 x}{\sin^3 x}$

5. $\lim_{t\to\pi}\frac{\cos\frac{\pi}{2}}{\pi - x}$

6. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

7. $\lim_{x\to 0}\frac{\sin 3x \sin 2x}{x\sin 4x}$

8. $\lim_{x\to+\infty}(1+\frac{1}{2x})^{3x}$

9. $\lim_{x\to-\infty}(1-\frac{3}{x})^{2x}$

10. $\lim_{t\to\infty}(e^x+x)^{1/x}$

11. $\lim_{x\to\infty}(1+\frac{1}{5x})^{-x}$

12. $\lim_{x\to+\infty}(1+x)^{1/x}$

13. $\lim_{x\to+\infty}(7-\frac{1}{x})^{6x}$