Day one.

IB Math 3 Indeterminate Limit and L'Hopital's rule

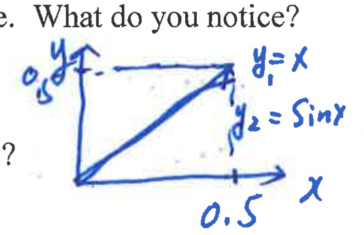
Name: key Period: ____

1. Revisiting a familiar trigonometric limit;

a. what is the $\lim_{x\to 0}\frac{\sin x}{x}$?

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

b. Sketch $y = x$ and $y = \sin x$ in the window $x:[0,0.5]$ by $y:[0,0.5]$ in radian mode. What do you notice?



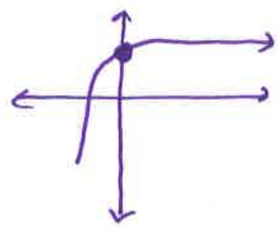
c. How does the graphs of $y = x$ and $y = \sin x$ help confirm the answer of $\lim_{x\to 0}\frac{\sin x}{x}$?

$$\lim_{x\to 0}\frac{\sin x}{x} \approx 1.$$

4. When we substitute 0 for x in the $\lim_{x\to 0}\frac{\sin x}{x}$, we are trying to find the limit of $\frac{0}{0}$. This is an example of an indeterminate limit. Some of the indeterminate limits are in the form of $\lim\frac{0}{0}$, $\lim\frac{\infty}{\infty}$, and $\lim 0\cdot\infty$.

a. Explain why $\lim_{x\to 1}\frac{2x^3+7x^2-9}{x^3+3x^2-4}$ is an example of an indeterminate limit.

$$\lim_{x\to 1}\frac{2(1)+7(1)-9}{(1)^3+3(1)^2-4} = \frac{0}{0} \Rightarrow \text{Indeterminate.}$$



b. Find the limit by graphing. Sketch your graph.

$$\lim_{x\to 1}\frac{2x^3+7x^2-9}{x^3+3x^2-4} = \boxed{2.22} \Leftarrow$$

x	.99	1	1.01
$\frac{f(x)}{g(x)} = y$	2.2226	undef	2.2219

≈ 2.22.

c. Let $f(x) = 2x^3+7x^2-9$ and $g(x) = x^3+3x^2-4$. Graph $f(x)$ and $g(x)$ in the window $x:[0.75,1.25]$ by $y:[-3,3]$. Compare the graphs of $f(x)$ and $g(x)$ at $x = 1$. Compare their slopes. Do they have the same slope? I one steeper than the other?

d. Find $f'(x)$ and $g'(x)$ and compute $\lim_{x\to 1} f'(x)$ and $\lim_{x\to 1} g'(x)$.

$f'(x) = 6x^2 + 14x \rightarrow \lim_{x\to 1} f'(x) = 6 + 14 = 20$

$g'(x) = 3x^2 + 6x \rightarrow \lim_{x\to 1} g'(x) = 3 + 6 = 9$

e. How do the limits of $f'(x)$ and $g'(x)$ compare to $\lim_{x\to 1}\frac{2x^3+7x^2-9}{x^3+3x^2-4}$

$$\lim_{x\to 1}\frac{f(x)}{g(x)} \quad \text{vs} \quad \lim_{x\to 1}\frac{f'(x)}{g'(x)} = \frac{20}{9}$$

≈ 2.22 $= 2.\bar{2}$

Day one

l'Hôpital's rule:

Theorem: Let $f(x)$ and $g(x)$ be the functions that are differentiable on an open interval (a,b) containing c. Assum that $g'(x) \neq 0$ for all x in (a,b). If the limit of $\frac{f(x)}{g(x)}$ as x approaches c produces the indeterminate form $\lim \frac{0}{0}$, $\lim \frac{\infty}{\infty}$, and $\lim 0 \cdot \infty$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Proof: Suppose $f(x)$ and $g(x)$ are differentiable function for $x \in R$.

and $\underline{f(c) = g(c) = 0}$. $\left(\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0} \right)$

$f(x) \approx L(x) = f'(c)(x-c) + f(c)$

$g(x) \approx G(x) = g'(c)(x-c) + g(c)$

$$\Rightarrow \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{L(x)}{G(x)} = \lim_{x \to c} \frac{f'(c)(x-c) + \overset{0}{\cancel{f(c)}}}{g'(c)(x-c) + \underset{0}{\cancel{g(c)}}} = \lim_{x \to c} \frac{f'(c)\cancel{(x-c)}}{g'(c)\cancel{(x-c)}}$$

$$\therefore \boxed{\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}} = \frac{f'(c)}{g'(c)}$$

Example 1) $\lim_{x \to 0} \frac{\sin x}{x}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = \boxed{1}$$

Example 2) $\lim_{x \to \infty} \frac{2x^2 - 3x + 1}{3x^2 + 5x - 2}$ $\left(\frac{\infty}{\infty}\right)$

$$= \lim_{x \to \infty} \frac{4x - 3}{6x + 5} \left(\frac{\infty}{\infty}\right) = \lim_{x \to \infty} \frac{4}{6} = \boxed{\frac{2}{3}}$$

Example 3) $\lim_{x \to 0} \frac{(1 - \cos x)\sin 4x}{x^3 \cos x}$

$$= \boxed{\lim_{x \to 0} \frac{(1 - \cos x)}{x^2}} \left(\frac{0}{0}\right) \lim_{x \to 0} \frac{\sin 4x}{x} \left(\frac{0}{0}\right) \cdot \lim_{x \to 0} \frac{1}{\cos x} \to 1$$

$$= \lim_{x \to 0} \frac{\sin x}{2x} \left(\frac{0}{0}\right) \lim_{x \to 0} \frac{4\cos 4x}{1} \cdot 1$$

$$= \lim_{x \to 0} \frac{\cos x}{2} \cdot 4 \cdot 1 = \frac{1}{2} \cdot 4 \cdot 1 = \boxed{2}$$

Example 4) $\lim_{x \to \infty} x \sin \frac{1}{x}$

$\left(\infty \cdot \sin \frac{1}{\infty}\right) \Rightarrow (\infty \cdot 0)$

$$\left(\begin{array}{l} u = \frac{1}{x} \Rightarrow x = \frac{1}{u} \\ x \to \infty \Rightarrow u \to 0 \end{array} \right) \Rightarrow \lim_{u \to 0} \frac{1}{u} \sin u = \lim_{u \to 0} \frac{\sin u}{u} \left(\frac{0}{0}\right) = \lim_{u \to 0} \frac{\cos u}{1}$$