(3) Linear Equations

$$\frac{dy}{dx} + Py = Q \quad (y' + Py = Q)$$

$\Rightarrow$ Multiply / use integrating factor $\boxed{I(x) = e^{\int P\,dx}}$

$$\Rightarrow e^{\int P dx} \cdot \frac{dy}{dx} + \underbrace{e^{\int P dx} \cdot P}_{I'(x)} \cdot y = e^{\int P dx} \cdot Q$$

$$\Rightarrow I(x) \cdot \frac{dy}{dx} + \frac{dI(x)}{dx} \cdot y = e^{\int P dx} \cdot Q$$

$$\Rightarrow \frac{d}{dx} I(x) \cdot y = I(x) \cdot Q$$

$$\Rightarrow \int \frac{d}{dx} I(x) \cdot y = \int I(x) \cdot Q$$

$$\Rightarrow y = \frac{1}{I(x)} \int I(x) \cdot Q\,dx \quad \text{where } I(x) = e^{\int P \cdot dx}$$

ex) $\frac{dy}{dx} + y = e^x \Rightarrow e^x \frac{dy}{dx} + e^x y = e^{2x}$

$\Rightarrow \int \frac{d}{dx}(e^x) \cdot y = \int e^{2x} \cdot dx$

$P = 1 \quad I(x) = e^{\int dx} = e^x$

$e^x \cdot y = \frac{1}{2} e^{2x}$

$y = \frac{1}{2} e^x + C$

ex) $x \frac{dy}{dx} - 3y = x^2 \Rightarrow \frac{dy}{dx} - \frac{3}{x} y = x$

$P = -\frac{3}{x} \quad I(x) = e^{\int -3/x\,dx} = e^{-3\ln x} = e^{\ln x^{-3}}$

$I(x) = \boxed{x^{-3}}$

$x^{-3} \frac{dy}{dx} - \frac{3}{x^4} \cdot y = x^{-2}$

$\int \frac{d}{dx} x^{-3} \cdot y = \int x^{-2} dx \Rightarrow y = x^3 \int x^{-2} dx$

$= x^3 \frac{-1}{x} = -x^3 + C$

$\boxed{y = -x^3 + C}$