IB Math HL 2 Mathematical Induction Exit Slip Name________________________

Problems 1 – 4: Use the Principle of Mathematical Induction to prove each statement.

1. If a sequence is defined by $u_1 = 2$ and $u_{n+1} = \dfrac{u_n}{2(n+1)}$ for all $n \in \mathbb{Z}^+$, then $u_n = \dfrac{2^{2-n}}{n!}$.

$P(n)$ is $u_n = \dfrac{2^{2-n}}{n!}$

$P(1)$: $u_1 = 2$ $\quad \dfrac{2^{2-1}}{1!} = \dfrac{2}{1} = 2$ ✓ So $P(1)$ is true

Assume $P(k)$ is true. Then $u_k = \dfrac{2^{2-k}}{k!}$

Consider $P(k+1)$

$$u_{k+1} = \frac{u_k}{2(k+1)}$$
$$= \frac{2^{2-k}}{k!} \cdot \frac{1}{2(k+1)}$$
$$= \frac{2^{2-k} \cdot 2^{-1}}{k!(k+1)} = \frac{2^{2-(k+1)}}{(k+1)!} \checkmark$$

So $P(k+1)$ is true.

$\therefore$ If $P(k)$ is true then $P(k+1)$ is true. Since $P(1)$ is true then $P(n)$ is true for all $n \in \mathbb{Z}^+$

2. $\cos x \times \cos 2x \times \cos 4x \times \cos 8x \times \ldots \times \cos\left(2^{n-1}x\right) = \dfrac{\sin\left(2^n x\right)}{2^n \times \sin x}$ for all $n \in \mathbb{Z}^+$.

$P(n)$ is $\cos x \cdot \cos 2x \cdot \ldots \cdot \cos(2^{n-1}x) = \dfrac{\sin(2^n x)}{2^n \cdot \sin x}$

$P(1)$: $\cos x \stackrel{?}{=} \dfrac{\sin 2x}{2\sin x} = \dfrac{2\sin x \cos x}{2 \sin x} = \cos x$ ✓ So $P(1)$ is true.

Assume $P(k)$ is true. Then $\cos x \cdot \cos 2x \cdot \ldots \cdot \cos(2^{k-1}x) = \dfrac{\sin 2^k x}{2^k \sin x}$

Consider $P(k+1)$.

Goal $\dfrac{\sin(2^{k+1}x)}{2^{k+1}\sin x}$

$$\cos x \cdot \cos 2x \cdot \ldots \cdot \cos(2^{k-1}x) \cdot \cos(2^k x)$$
$$= \frac{\sin(2^k x)}{2^k \sin x} \cdot \cos(2^k x)$$
$$= \frac{\sin(2^k x)\cos(2^k x)}{2^k \sin x}$$
$$= \frac{\frac{1}{2}\sin(2 \cdot 2^k x)}{2^k \sin x}$$
$$= \frac{\sin(2^{k+1}x)}{2^{k+1}\sin x} \checkmark$$

So $P(k+1)$ is true.

$\therefore$ If $P(k)$ is true ...

3. $7^n + 3^n + 2$ is divisible by 4 for all $n \in \mathbb{N}$.

$P(n)$ is $7^n + 3^n + 2$ is divisible by 4.

$P(0)$: $7^0 + 3^0 + 2 = 4 = 4 \cdot 1$ So $P(1)$ is true

assume $P(k)$ is true. Then $7^k + 3^k + 2 = 4A$ for some $A \in \mathbb{Z}^+$

Consider $P(k+1)$.

$$\begin{aligned} 7^{k+1} + 3^{k+1} + 2 &= 7^k \cdot 7 + 3^{k+1} + 2 \\ &= (4A - 2 - 3^k) \cdot 7 + 3^{k+1} + 2 \\ &= 4(7A) - 14 - 7 \cdot 3^k + 3 \cdot 3^k + 2 \\ &= 4(7A) - 12 - 4(3^k) \\ &= 4(7A - 3 - 3^k) \end{aligned}$$

$\therefore$ If $P(k)$ is true ...

So $P(k+1)$ is true.

4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{Z}^+$.

$P(n)$ is $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

$P(1)$: $\sum_{i=1}^{1} i^3 = 1$ $\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$ ✓ So $P(1)$ is true.

Assume $P(k)$ is true. Then $\sum_{k=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$

Consider $P(k+1)$

Goal $\frac{(k+1)^2(k+2)^2}{4}$

$$\begin{aligned} \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= (k+1)^2 \left[\frac{k^2}{4} + \frac{4(k+1)}{4} \right] \\ &= (k+1)^2 \left(\frac{k^2 + 4k + 4}{4} \right) \\ &= \frac{(k+1)^2(k+2)^2}{4} \checkmark \end{aligned}$$

$\therefore$ If $P(k)$ is true ...

so $P(k+1)$ is true.

5. Consider the function $f(x)=e^{-x}(x+2)$.

a. Find i. $f'(x)$ ii. $f''(x)$ iii. $f'''(x)$ iv. $f^{(4)}(x)$

$f'(x) = -e^{-x}(x+2) + e^{-x}$
$= -e^{-x}(x+1)$

$f''(x) = e^{-x}(x+1) - e^{-x}\cdot 1$
$= e^{-x}x$

$f'''(x) = -e^{-x}\cdot x + e^{-x}$
$= -e^{-x}(x-1)$

$f^{(4)}(x) = e^{-x}(x-1) - e^{-x}\cdot 1$
$= e^{-x}(x-2)$

b. Conjecture a formula for $f^{(n)}(x),\ n \in \mathbb{Z}^+$.

$f^{(n)}(x) = (-1)^n e^{-x}(x+2-n)$

c. Use the principle of mathematical induction to prove your conjecture.

$P(n)$ is $f^{(n)}(x) = (-1)^n e^{-x}(x+2-n)$

$P(1)$: $f'(x) = -e^{-x}(x+1)$ $\quad (-1)^1 e^{-x}(x+2-1)$ So $P(1)$ is true

$-e^{-x}(x+1)$ ✓

Assume $P(k)$ is true. Then $f^{(k)}(x) = (-1)^k e^{-x}(x+2-k)$

Consider $P(k+1)$

$f^{(k+1)}(x) = (-1)^k(-1)e^{-x}(x+2-k) + (-1)^k e^{-x}(1)$
$= (-1)^{k+1} e^{-x}(x+2-k-1)$
$= (-1)^{k+1} e^{-x}(x+2-(k+1))$

So $P(k+1)$ is true.

$\therefore$ If $P(k)$ is true ...