Differential Equations : LB Exam Questions #2 Key ①

#1. a) $y = x \cdot v \Rightarrow \frac{dy}{dx} = \boxed{v + x\frac{dv}{dx} = F(v)}$

$$\Rightarrow x \cdot \frac{dv}{dx} = F(v) - v$$

$$\Rightarrow \frac{x}{dx} = \frac{F(v) - v}{dv}$$

$$\Rightarrow \boxed{\frac{dx}{x} = \frac{dv}{F(v) - v}}$$

b) $X = x - 1$ $\quad Y = y - 2$

$dX = dx$ $\quad dY = dy$

$\boxed{x = X + 1}$ $\quad \boxed{y = Y + 2}$

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{x + 3y - 7}{3x - y - 1} = \frac{(X+1) + 3(Y+2) - 7}{3(X+1) - (Y+2) - 1}$$

$$= \frac{X + 1 + 3Y + 6 - 7}{3X + 3 - Y - 2 - 1} = \frac{X + 3Y}{3X - Y} \quad \begin{matrix} \div X \\ \div X. \end{matrix}$$

$$\frac{dY}{dX} = \frac{1 + 3 \cdot \frac{Y}{X}}{3 - \frac{Y}{X}} \Leftarrow \left(V = \frac{Y}{X}. \quad \frac{dY}{dX} = V + X\frac{dV}{dX}. \right)$$

$$V + X\frac{dV}{dX} = \frac{1 + 3V}{3 - V} \Rightarrow \left(V + X\frac{dV}{dX}\right)(3 - V) = 1 + 3V$$

$$\Rightarrow \cancel{3V} - V^2 + 3X\frac{dV}{dX} - XV\frac{dV}{dX} = 1 + \cancel{3V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1 + V^2}{3 - V} \Rightarrow \int \frac{dX}{X} = \int \frac{3 - V}{1 + V^2}\, dV$$

$\Rightarrow$

$$\int \frac{dx}{x} = \int \frac{3}{1+V^2} - \int \frac{V}{1+V^2}\, dV$$

$$\Rightarrow \boxed{\ln X = 3\arctan\left(\frac{Y}{X}\right) - \frac{1}{2}\ln\left(1+\left(\frac{Y}{X}\right)^2\right) + C}$$

$$\boxed{Y = y-2 \qquad X = x-1}$$

$$\Rightarrow \boxed{\ln(x-1) = 3\arctan\left(\frac{y-2}{x-1}\right) - \frac{1}{2}\ln\left(1+\left(\frac{y-2}{x-1}\right)^2\right) + C}$$

\# 2. $\frac{dy}{dx} = \cos x \cot^2 y \Rightarrow \frac{dy}{\cot^2 y} = \cos x\, dx \Rightarrow \int \tan^2 y\, dy = \int \cos x\, dx$

$$\Rightarrow \int (\sec^2 y - 1)\, dy = \int \cos x\, dx$$

$$\Rightarrow \tan y - y = \sin x + C \qquad \Leftarrow x = \pi \quad y = \frac{\pi}{4}$$

$$\tan\frac{\pi}{4} - \frac{\pi}{4} = \sin\pi + C \qquad \Rightarrow C = 1 - \frac{\pi}{4}.$$

$$\boxed{\tan y - y = \sin x + \left[1 - \frac{\pi}{4}\right]} \Leftarrow \text{cannot solve for } y.$$

\# 3. $\frac{x^2}{x^2}\frac{dy}{dx} = \frac{y^2 + xy + 4x^2}{x^2}$

Sub.

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x} + 4 \qquad \Leftarrow \boxed{V = \frac{y}{x} \quad y = x\cdot V \quad \frac{dy}{dx} = V + x\cdot\frac{dV}{dx}}$$

$$\cancel{V} + x\cdot\frac{dV}{dx} = V^2 + \cancel{V} + 4$$

$$\frac{x}{dx} = \frac{V^2+4}{dV} \Rightarrow \int \frac{dx}{x} = \int \frac{dV}{V^2+4} \Rightarrow \ln x = \frac{1}{2}\arctan\left(\frac{V}{2}\right) + C$$

$\Rightarrow$

$\ln x = \frac{1}{2} \arctan\left(\frac{y}{2x}\right) + c$ $\qquad x=1 \quad y=2$

$0 = \frac{1}{2} \arctan(1) + c$ $\qquad C = -\frac{1}{2} \cdot \frac{\pi}{4} = -\frac{\pi}{8}$

$\Rightarrow \ln x = \frac{1}{2} \arctan\left(\frac{y}{2x}\right) - \frac{\pi}{8}.$

$\arctan\left(\frac{y}{2x}\right) = \left(\ln x + \frac{\pi}{8}\right) \cdot 2$

$\left(\frac{y}{2x}\right) = \tan\left(2 \ln x + \frac{\pi}{4}\right)$

$\boxed{y = 2x \tan\left(2 \ln x + \frac{\pi}{4}\right)}$

#4.

$\frac{dy}{dx} = 2e^x + y \tan x$

$\frac{dy}{dx} - \tan x \, y = 2e^x. \quad \Rightarrow I(x) = e^{\int -\tan x \, dx} = e^{\ln \cos x} = \cos x$

$\cos x \cdot \frac{dy}{dx} - (\tan x \cdot \cos x) \cdot y = 2e^x \cdot \cos x$

$\underbrace{\cos x \cdot \frac{dy}{dx} - \sin x \cdot y} = 2e^x \cdot \cos x$

$\int \frac{d}{dx}[\cos x \cdot y] = \int 2e^x \cdot \cos x \, dx$

u	dv
$2e^x$	$\cos x$
$2e^x$ $(+)$	$\sin x$
$2e^x$ $(-)$	$-\cos x$
$(+)$	

$\cos x \cdot y = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C$

$x = 0 \quad y = 1$

$\boxed{C = 1 - \frac{1}{2} = \frac{1}{2}}.$

$\int \cancel{2}e^x \cdot \cos x \, dx = 2e^x \sin x + 2e^x \cos x - \cancel{2}\int e^x \cos x \, dx$

$\Rightarrow$

$$\cos x \cdot y = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + \frac{1}{2}$$

$$\boxed{y = \frac{1}{2} e^x \tan x + \frac{1}{2} e^x + \frac{1}{2} \sec x}$$

#5 . $$\frac{x^2 \frac{dy}{dx}}{x^2} = \frac{y^2 + 3xy + 2x^2}{x^2}$$

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{3y}{x} + 2 \qquad \Leftarrow V = \frac{y}{x} \qquad \frac{dy}{dx} = V + x \cdot \frac{dv}{dx}$$

$$V + x \cdot \frac{dv}{dx} = V^2 + 3V + 2$$

$$x \cdot \frac{dv}{dx} = V^2 + 2V + 2$$

$$\int \frac{dx}{x} = \frac{dv}{V^2 + 2V + 1 + 1} = \int \frac{dv}{(V+1)^2 + 1}$$

$$\ln x = \arctan(V+1) + C$$

$$\ln x = \arctan\left(\frac{x}{y} + 1\right) + C \qquad \Leftarrow x = 1 \quad y = -1$$

$$0 = \arctan(-1+1) + C \Rightarrow C = 0$$

$$\ln x = \arctan\left(\frac{x}{y} + 1\right)$$

$$\Rightarrow \left(\frac{y}{x} + 1\right) = \tan(\ln x)$$

$$\Rightarrow \frac{y}{x} = \tan(\ln x) - 1 \Rightarrow \boxed{y = x \tan(\ln x) - x}$$

\# 6

a) $(\ln x)\dfrac{dy}{dx} + \dfrac{2}{x}\cdot y = \dfrac{2x-1}{\ln x}$ $\quad x > 1$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{2}{x(\ln x)}y = \dfrac{2x-1}{(\ln x)^2}$ $\quad (y' + P\cdot y = Q)$

$\Rightarrow \underbrace{(\ln x)^2 \cdot \dfrac{dy}{dx} + \left(\dfrac{2\ln x}{x}\right)(y)} = 2x-1$ $\quad P = \dfrac{2}{x(\ln x)} \quad Q = \dfrac{2x-1}{(\ln x)^2}$

$I = e^{\int \frac{2}{x(\ln x)}dx} = e^{2\ln(\ln x)^2}$

$= (\ln x)^2$

$\int \dfrac{d}{dx}\left[y\cdot(\ln x)^2\right] = \int 2x-1\,dx$

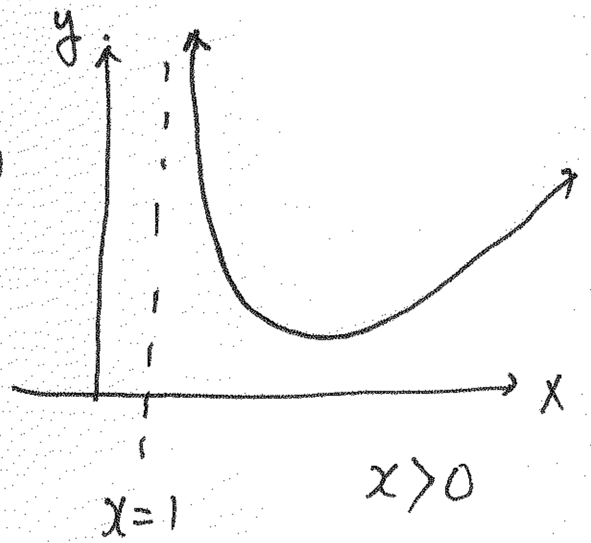
$y\cdot(\ln x)^2 = x^2 - x + C$

$\boxed{y = \dfrac{x^2 - x + C}{(\ln x)^2}}$ $\Leftarrow$ $x = e \quad y = e^2$

b) $e^2 = \dfrac{e^2 - e + c}{(\ln e)^2} \Rightarrow c = e$

$\Rightarrow \boxed{y = \dfrac{x^2 - x + e}{(\ln x)^2}}$

c)



$x > 0$

Vertical Asymptote.

$\ln x = 0$

$x = 1$