# More Substitution

1. $\int x^2\sqrt{x+1}\,dx$ $\qquad u = x+1$

$du = dx$

$\int (u-1)^2\sqrt{u}\,du$

$\int (u^2-2u+1)u^{\frac{1}{2}}\,du$

$\int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

$\frac{2}{7}u^{\frac{7}{2}} - 2\cdot\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C$

$$\boxed{\frac{2}{7}(x+1)^{\frac{7}{2}} - \frac{4}{5}(x+1)^{\frac{5}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + C}$$

2. $\int \frac{x+1}{1+x^2}\,dx = \int \frac{x}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx$ $\qquad x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$u = 1+x^2$

$du = 2x\,dx$

$\int \frac{1}{2}\cdot\frac{1}{u}\,du + \int \frac{\cancel{\sec^2}\theta\,d\theta}{\cancel{1+\tan^2\theta}}$

$\frac{1}{2}\ln u + \int d\theta$

$\frac{1}{2}\ln(1+x^2) + \theta + C$

$$\boxed{\frac{1}{2}\ln(1+x^2) + \arctan x + C}$$

3. $\int \frac{1}{\sqrt{9-4x^2}}\,dx$ $\qquad 2x = 3\sin\theta \longrightarrow \theta = \arcsin\frac{2x}{3}$

$2dx = 3\cos\theta\,d\theta \qquad (2x)^2 = 4x^2 = 9\sin^2\theta$

$\int \frac{\frac{3}{2}\cos\theta\,d\theta}{\sqrt{9-(3\sin\theta)^2}}$

$\int \frac{3}{2}\cdot\frac{\cos\theta\,d\theta}{\sqrt{9(1-\sin^2\theta)}}$

$\int \frac{3}{2}\cdot\frac{1}{3}\,\frac{\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}}$

$\int \frac{1}{2}\,d\theta$

$\frac{1}{2}\theta + C$

$$\boxed{\frac{1}{2}\arcsin\frac{2x}{3} + C}$$

4. $\displaystyle\int \frac{x}{\sqrt{x+4}}\,dx$ $\qquad u = x+4$

$\qquad du = dx$

$$\int \frac{u-4}{u^{1/2}}\,du$$

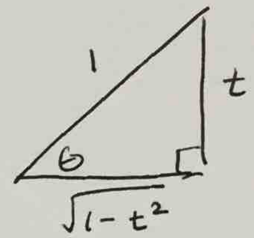
$$\int \left(u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}\right) du$$

$$\frac{2}{3}u^{\frac{3}{2}} - 4\cdot 2\,u^{\frac{1}{2}} + C$$

$$\boxed{\frac{2}{3}(x+4)^{\frac{3}{2}} - 8\sqrt{x+4} + C}$$

5. $\displaystyle\int_0^{\sqrt{3}/2} \frac{t^2}{(1-t^2)^{3/2}}\,dt$ $\qquad t = \sin\theta$

$\qquad dt = \cos\theta\, d\theta$




$$\int \frac{t^2}{(1-t^2)^{3/2}}\,dt = \int \frac{\sin^2\theta}{(1-\sin^2\theta)^{3/2}}\cdot \cos\theta\, d\theta$$

$$= \int \frac{\sin^2\theta\cos\theta\, d\theta}{(\cos^2\theta)^{3/2}}$$

$$= \int \frac{\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta}$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$$

$$= \int \tan^2\theta\, d\theta$$

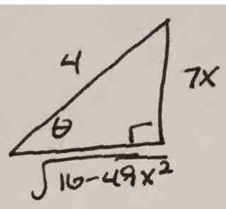
$$= \int (\sec^2\theta - 1)\,d\theta$$

$$= \tan\theta - \theta + C = \frac{t}{\sqrt{1-t^2}} - \arcsin t + C$$

$$\int_0^{\sqrt{3}/2} \frac{t^2}{(1-t^2)^{3/2}} = \left[\frac{t}{(1-t^2)^{1/2}} - \arcsin t\right]_0^{\sqrt{3}/2} = \left(\frac{\frac{\sqrt{3}}{2}}{\left(1-\frac{3}{4}\right)^{1/2}} - \arcsin\frac{\sqrt{3}}{2}\right) - (0 - \arcsin 0)$$

$$= \left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} - \frac{\pi}{3}\right) - 0 = \boxed{\sqrt{3} - \frac{\pi}{3}}$$

6. $\int \sqrt{16-49x^2}\,dx$ $\qquad 7x = 4\sin\theta$ $\qquad 7dx = 4\cos\theta\, d\theta$




$\int \sqrt{16-16\sin^2\theta} \cdot \frac{4}{7}\cos\theta\, d\theta$

$\int 4\sqrt{1-\sin^2\theta} \cdot \frac{4}{7}\cos\theta\, d\theta$

$\int \frac{16}{7}\cos^2\theta\, d\theta$

$\int \frac{16}{7}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$

$\frac{16}{7}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + C$

$\frac{8}{7}\arcsin\frac{7x}{4} + \frac{4}{7}(2\sin\theta\cos\theta) + C$

$\frac{8}{7}\arcsin\frac{7x}{4} + \frac{8}{7}\left(\frac{7x}{4}\right)\left(\frac{\sqrt{16-49x^2}}{4}\right) + C$

$\boxed{\frac{8}{7}\arcsin\frac{7x}{4} + \frac{x}{2}\sqrt{16-49x^2} + C}$

7. $\int e^{2x}\sqrt{1+e^{2x}}\,dx$ $\qquad u = 1+e^{2x}$ $\qquad du = 2e^{2x}dx$

$\int \frac{1}{2}u^{\frac{1}{2}}du$

$\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \boxed{\frac{1}{3}(1+e^{2x})^{3/2} + C}$

8. $\int \frac{\ln(x+1)}{x+1}dx$ $\qquad u = \ln(x+1)$ $\qquad du = \frac{1}{x+1}dx$

$\int u\,du$

$\frac{1}{2}u^2 + C = \boxed{\frac{1}{2}(\ln(x+1))^2 + C}$

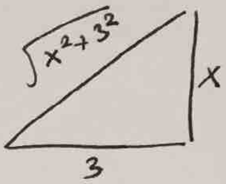
9. $\int \frac{2x-5}{4x^2}dx = \int\left(\frac{2x}{4x^2} - \frac{5}{4x^2}\right)dx = \int\left(\frac{1}{2x} - \frac{5}{4}x^{-2}\right)dx$

$= \frac{1}{2}\ln x + \frac{5}{4}x^{-1} + C = \boxed{\frac{1}{2}\ln x + \frac{5}{4x} + C}$

10. $\displaystyle\int_0^3 \frac{x^3}{\sqrt{(x^2+9)}}\,dx$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$




$$\int \frac{x^3}{\sqrt{(x^2+9)}}\,dx = \int \frac{27\tan^3\theta}{\sqrt{9\tan^2\theta+9}} \cdot 3\sec^2\theta\, d\theta$$

$$= \int \frac{27\tan^3\theta}{3\sec\theta} \cdot 3\sec^2\theta\, d\theta$$

$$= \int 27\tan^3\theta \sec\theta\, d\theta$$

$$= \int 27(\sec^2\theta - 1)\tan\theta\sec\theta\, d\theta$$

$u = \sec\theta$

$du = \sec\theta\tan\theta\, d\theta$

$$= \int 27(u^2-1)\,du$$

$$= 27\left(\tfrac{1}{3}u^3 - u\right) + C$$

$$= 9\sec^3\theta - 27\sec\theta + C$$

$$= 9\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 27\left(\frac{\sqrt{x^2+9}}{3}\right) + C$$

$$= \tfrac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C$$

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = \left[\tfrac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9}\right]_0^3$$

$$= \left(\tfrac{1}{3}(18)^{3/2} - 9\sqrt{18}\right) - \left(\tfrac{1}{3}(9)^{3/2} - 9\sqrt{9}\right)$$

$$= \tfrac{1}{3}\cdot 18\sqrt{18} - 9\cdot 3\sqrt{2} - \tfrac{1}{3}\cdot 27 + 27$$

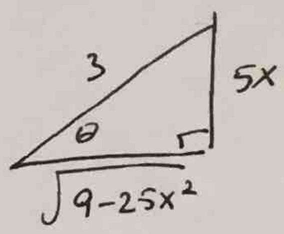
$$= 6\cdot 3\sqrt{2} - 27\sqrt{2} + 18$$

$$= \boxed{18 - 9\sqrt{2}}$$

11. $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\,dx$

$5x = 3\sin\theta$

$5\,dx = 3\cos\theta\,d\theta$




$$\int \sqrt{9-25x^2}\,dx = \int \sqrt{9-9\sin^2\theta}\cdot \frac{3}{5}\cos\theta\,d\theta$$

$$= \int 3\cos\theta \cdot \frac{3}{5}\cos\theta\,d\theta$$

$$= \int \frac{9}{5}\cos^2\theta\,d\theta$$

$$= \int \frac{9}{5}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{9}{5}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + C$$

$$= \frac{9}{10}\theta + \frac{9}{20}\left(2\sin\theta\cos\theta\right) + C$$

$$= \frac{9}{10}\left(\arcsin\frac{5x}{3}\right) + \frac{9}{10}\left(\frac{5x}{3}\right)\left(\frac{\sqrt{9-25x^2}}{3}\right) + C$$

$$= \frac{9}{10}\arcsin\frac{5x}{3} + \frac{x}{2}\sqrt{9-25x^2} + C$$

$$\int_0^{3/5}\sqrt{9-25x^2}\,dx = \left[\frac{9}{10}\arcsin\frac{5x}{3} + \frac{x}{2}\sqrt{9-25x^2}\,\right]_0^{3/5}$$

$$= \left(\frac{9}{10}\arcsin 1 + \frac{3}{10}\sqrt{9-9}\right) - \left(\frac{9}{10}\arcsin 0 + 0\right)$$

$$= \frac{9}{10}\cdot\frac{\pi}{2} + 0 - 0$$

$$= \boxed{\frac{9\pi}{20}}$$

12. $\int e^x \sqrt{1-e^{2x}}\,dx$ $\qquad u = e^x$

$\qquad du = e^x\,dx$

$\int \sqrt{1-u^2}\,du$ $\qquad u = \sin\theta$

$\qquad du = \cos\theta\,d\theta$

$\int \sqrt{1-\sin^2\theta} \cdot \cos\theta\,d\theta$

$\int \cos^2\theta\,d\theta = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$

$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$

$= \frac{1}{2}\theta + \frac{1}{4}(2\sin\theta\cos\theta) + C$

$= \frac{1}{2}\arcsin u + \frac{1}{2}\cdot u \cdot \sqrt{1-u^2} + C$

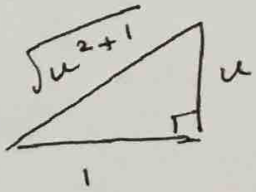
$= \boxed{\frac{1}{2}\arcsin e^x + \frac{1}{2}e^x\sqrt{1-e^{2x}} + C}$

13. $\int (x+1)\sqrt{x^2+2x+2}\,dx$ $\qquad u = x+1$

$\qquad du = dx$

$\int (x+1)\sqrt{(x+1)^2+1}\,dx$ $\qquad u = \tan\theta$

$\int u\sqrt{u^2+1}\,du$ $\qquad du = \sec^2\theta\,d\theta$




$\sec\theta = \sqrt{u^2+1}$

$\int \tan\theta\sqrt{\tan^2\theta+1} \cdot \sec^2\theta\,d\theta$

$\int \tan\theta \cdot \sec\theta \cdot \sec^2\theta\,d\theta$ $\qquad w = \sec\theta$

$\qquad dw = \sec\theta\tan\theta\,d\theta$

$\int \tan\theta\sec\theta\sec^2\theta\,d\theta$

$\int w^2\,dw$ $\qquad \frac{1}{3}\left(\sqrt{u^2+1}\right)^3 + C$

$\frac{1}{3}w^3 + C$ $\qquad \boxed{\frac{1}{3}\left((x+1)^2+1\right)^{3/2} + C}$

$\frac{1}{3}\sec^3\theta + C$

Yes, you could have used regular u-substitution instead, but wasn't that fun? :)