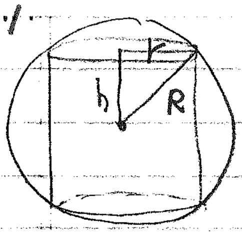
①

\#1 Optimization



$r^2+h^2=R^2 \Rightarrow h=\sqrt{R^2-r^2}$

$S.A_{\text{cylinder}} = 2\pi r(2h) = 4\pi rh = 4\pi r\sqrt{R^2-r^2}$

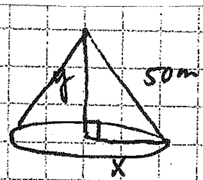
$SA' = 4\pi\left[r'\sqrt{R^2-r^2} + r\frac{1}{2}(R^2-r^2)^{-\frac{1}{2}}(-2r)\right]$

$= 4\pi\left[\dfrac{R^2-2r^2}{\sqrt{R^2-r^2}}\right] = 0$

$\boxed{r=\frac{R}{\sqrt{2}}}$ $\quad h=\sqrt{R^2-\frac{R^2}{2}} = \boxed{\frac{R}{\sqrt{2}}=h}$

$H = 2h = \dfrac{2R}{\sqrt{2}}$

\#2

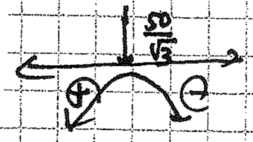


$x^2+y^2=50^2 \Rightarrow x^2 = 50^2-y^2$

$V=\frac{1}{3}\pi x^2\cdot y = \frac{1}{3}\pi(50^2-y^2)y = \frac{1}{3}\pi(50^2y-y^3)$

$V' = \frac{1}{3}\pi[50^2-3y^2] = 0 \Rightarrow y^2 = \frac{50^2}{3}$

$y = \pm\sqrt{\frac{50^2}{3}} = \pm\frac{50}{\sqrt{3}}$



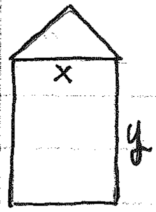
Max.

Sign diagram

$V=\frac{1}{3}\pi\left(50^2\left(\frac{50}{\sqrt{3}}\right) - \left(\frac{50}{\sqrt{3}}\right)^3\right)$

$=\frac{1}{3}\pi\left(\frac{50^3}{\sqrt{3}} - \frac{50^3}{3\sqrt{3}}\right)$

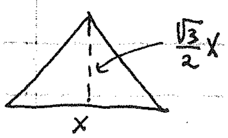
\#3.



$p = 3x+2y = 20 \Rightarrow y = \dfrac{20-3x}{2} \qquad 0<x<\frac{20}{3}$

Total Area $= xy + (\frac{1}{2})(x)(\frac{\sqrt{3}}{2}x)$

Total Area $= x\left[\dfrac{20-3x}{2}\right] + \left[\dfrac{\sqrt{3}x^2}{4}\right]$



For light Transmission $= k\left[20x - \frac{3}{2}x^2 + \left(\frac{\sqrt{3}x^2}{4}\right)\right]$

$= k\left[20x - \frac{3}{2}x^2 + \frac{\sqrt{3}}{4}x^2\right]$

$= k\left(20x - 3x^2 + \frac{\sqrt{3}}{4}x^2\right)$

$AT' = k\left(20-6x+\frac{\sqrt{3}}{2}x\right) = 0$

$x\left(\frac{\sqrt{3}}{2}-6\right) = -20$

$x = \boxed{\dfrac{20}{6-\frac{\sqrt{3}}{2}} \approx 3.8956 \approx 4\text{ft}}$

$y \approx \boxed{4.1566\text{ ft} \approx 4\text{ft}}$

(2)

4. a. $\theta r = 2\pi R$ and $h = \sqrt{r^2 - R^2}$.

$$\therefore V = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi \times \left(\frac{\theta r}{2\pi}\right)^2 \sqrt{r^2 - \left(\frac{\theta r}{2\pi}\right)^2} = \frac{1}{3}\pi \times \frac{\theta^2 r^2}{4\pi^2}\sqrt{\frac{4\pi^2 r^2 - \theta^2 r^2}{4\pi^2}} = \frac{\theta^2 r^3}{24\pi^2}\sqrt{4\pi^2 - \theta^2},\ 0 < \theta < 2\pi.$$

Note: the case where $\theta = 0, 2\pi \Rightarrow V = 0$.

b. $$\frac{dV}{d\theta} = \frac{2\theta r^3}{24\pi^2}\sqrt{4\pi^2 - \theta^2} + \frac{1}{2}\cdot\frac{\theta^2 r^3}{24\pi^2}\cdot\frac{-2\theta}{\sqrt{4\pi^2 - \theta^2}} = \frac{\theta r^3}{24\pi^2}\left[2\sqrt{4\pi^2 - \theta^2} - \frac{\theta^2}{\sqrt{4\pi^2 - \theta^2}}\right]$$

$$= \frac{\theta r^3}{24\pi^2}\left[\frac{8\pi^2 - 3\theta^2}{\sqrt{4\pi^2 - \theta^2}}\right].$$

c. Now, $\frac{dV}{d\theta} = 0 \Leftrightarrow \frac{\theta r^3}{24\pi^2}\left[\frac{8\pi^2 - 3\theta^2}{\sqrt{4\pi^2 - \theta^2}}\right] \Leftrightarrow \theta = 0$ or $\theta = \pm\pi\sqrt{\frac{8}{3}}$. As $\theta > 0$, only possible solution is $\theta = \pi\sqrt{\frac{8}{3}}$. Using a sign diagram of $\frac{dV}{d\theta}$ we have that V is a maximum when $\theta = \pi\sqrt{\frac{8}{3}}$.

Therefore, $k = \sqrt{\frac{8}{3}} = \frac{2\sqrt{6}}{3}$.

Related Rate

#1

Let x be the radius of the top circle of the body of water and y its height. The radius of the top circle is 20, the height of the cone is 40 ft. By similar right triangles,

$$\frac{20}{40} = \frac{x}{y}$$
$$x = \frac{1}{2}y$$

The volume of the body of water is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{2}y\right)^2(y) = \frac{1}{12}\pi y^3$$

Then,

$$\frac{dV}{dt} = \frac{1}{4}\pi y^2 \frac{dy}{dt}$$

When $y = 12$, $x = 6$ and $dV/dt = 80$, so

$$80 = \frac{1}{4}\pi(12)^2\frac{dy}{dt}$$
$$\frac{dy}{dt} = \frac{80(4)}{\pi(144)} \approx 0.71 \text{ ft/min}$$

2

At noon, the car is at the origin, while the truck is at (250, 0). At time t, the truck is at position $(250 - x, 0)$, while the car is at $(0, y)$. Let H be the distance between them. Also, we are given $dx/dt = 25$, $dy/dt = 50$, $x = 25t$ and $y = 50t$.

$$H^2 = (250 - x)^2 + y^2$$
$$H^2 = (250 - 25t)^2 + 50^2t^2$$
$$H^2 = 3{,}125t^2 - 12{,}500t + 62{,}500$$

a. $$2H\frac{dH}{dt} = 6{,}250t - 12{,}500$$
$$\frac{dH}{dt} = \frac{3{,}125t - 6{,}250}{\sqrt{3{,}125t^2 - 12{,}500t + 62{,}500}}$$

b. $dH/dt = 0$ when $3{,}125t - 6{,}250 = 0$ or when $t = 2$.

c. $$H^2 = 3{,}125(2)^2 - 12{,}500(2) + 62{,}500 = 50{,}000$$
$$H = 100\sqrt{5} \approx 224 \text{ mi (reject negative)}$$

#3

Note: 20 m is $20/1{,}000 = 1/50$ km. Let x be the distance between the race car and the finish line, then $dx/dt = -200$ km/h.

$$x = \frac{1}{50}\tan\theta$$
$$\frac{dx}{dt} = \frac{1}{50}\sec^2\theta\,\frac{d\theta}{dt}$$

We are given that $\frac{dx}{dt} = -200$.

When $\theta = 0$,

$\sec 0 = 1$:

$$-200 = \frac{1}{50}(1)^2\frac{d\theta}{dt}$$
$$-10{,}000 = \frac{d\theta}{dt}$$ This is rad/hr

The angle of the line of sight is changing at the rate of 2.78 rad/s.

Assume the ice is in the shape of a sphere of radius r.

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

With $r = 4$ and $\frac{dV}{dt} = -5$,

$$-5 = 4\pi(4)^2\frac{dr}{dt}$$
$$\frac{dr}{dt} = -\frac{5}{64\pi} \approx -0.025 \text{ in./min}$$

The radius is decreasing at the rate of 0.025 in./min.

The surface area is given by

$$S = 4\pi r^2$$
$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$$

With $r = 4$ and $\frac{dr}{dt} = \frac{-5}{64\pi}$,

$$\frac{dS}{dt} = 8\pi(4)\left(-\frac{5}{64\pi}\right) = -2.5 \text{ in.}^2\text{/min.}$$

The surface area is decreasing at the rate of 2.5 in.^{2}/min.