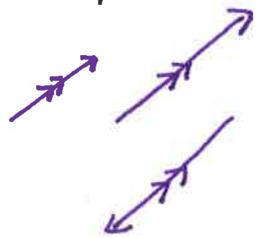
IB Math HL1: 14I Parallelism

1. Find r and s so that $\mathbf{a}$ and $\mathbf{b}$ are parallel if $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ r \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} s \\ 2 \\ -3 \end{pmatrix}$.

   Hint: **b** is a multiple of **a**.



$\vec{b} = m\vec{a}$

$m \in \mathbb{R}$

$\begin{pmatrix} s \\ 2 \\ -3 \end{pmatrix} = m \begin{pmatrix} 2 \\ -1 \\ r \end{pmatrix} \Rightarrow m = -2$

$(-3) = (-2)r \Rightarrow r = \frac{3}{2}$

$s = (2)\cdot(-2) \Rightarrow s = -4$

2. Given $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$,

a. Find a unit vector in the direction of **a**.

   Step 1: Find $|\mathbf{a}|$

   Step 2: Divide **a** by its own magnitude.

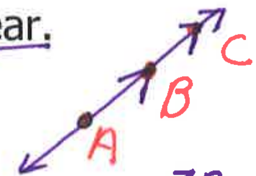
unit vector of $a = \frac{3}{\sqrt{10}} i - \frac{1}{\sqrt{10}} j$

b. Find a vector of length 4 in the direction of **a**.

$4(\text{unit vector of } a) = \frac{12}{\sqrt{10}} i - \frac{4}{\sqrt{10}} j$

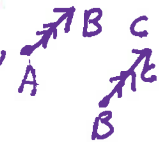
3. Show that A(-1, 2, 3), B(4, 0, -1), and C(14, -4, -9) are collinear.

Hint: Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$. What do you notice about these vectors?



$$\vec{AB} = \begin{pmatrix} 4-(-1) \\ 0-2 \\ -1-(3) \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix} = 5i - 2j - 4k.$$

$\Rightarrow (\vec{AB}) \cdot 2 = \vec{BC}$ show



$$\vec{BC} = \begin{pmatrix} 14-4 \\ -4-0 \\ -9-(-1) \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ -8 \end{pmatrix} = 10i - 4j - 8k.$$

$\vec{AB} \parallel \vec{BC}$

$\therefore$ A, B, & C are collinear.

4. Find a and b if K(1, -1, 0), L(4, -3, 7), and M(a, 2, b) are collinear.

$$\vec{KL} = \begin{pmatrix} 4-1 \\ -3-(-1) \\ 7-0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$$

$\Rightarrow \vec{KL} \parallel \vec{LM}$

$x: (3)\left(\frac{-5}{2}\right) = a-4$

$a = \frac{-7}{2}$

$$\vec{LM} = \begin{pmatrix} a-4 \\ 2-(-3) \\ b-7 \end{pmatrix} = \begin{pmatrix} a-4 \\ 5 \\ b-7 \end{pmatrix}$$

$y: (-2)m = 5$

$m = \frac{-5}{2}.$

$z: (7)\left(\frac{-5}{2}\right) = b-7$

$b = \frac{-21}{2}$

14J Dot Product

If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, the **Dot Product** of **v** and **w** is

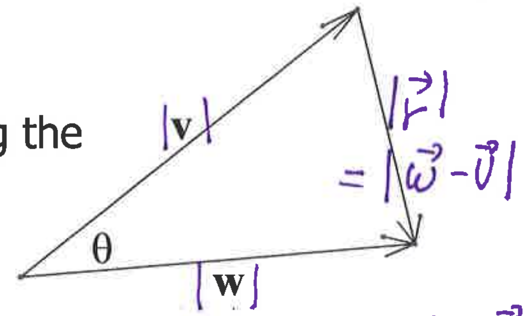
defined as $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

$\vec{v} \cdot \vec{w} \neq \vec{v} \times \vec{w}$

<u>Example 1</u> If $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$, find $\mathbf{v} \cdot \mathbf{w}$

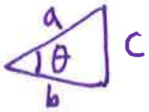
$$\vec{v} \cdot \vec{w} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix} = 15 - 2 - 6 = \boxed{7}$$

14J Applications of the Dot Product



$\vec{v} + \vec{r} = \vec{w} \Rightarrow \vec{r} = \vec{w} - \vec{v}$

Use the Law of Cosines to write an equation relating the magnitudes of these vectors. Solve for $\cos\theta$.



$\Rightarrow c^2 = a^2 + b^2 - 2ab\cdot\cos\theta$.

Where $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$

$(a-b)^2 = a^2 - 2ab + b^2$

$$|\vec{w} - \vec{v}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos\theta$$

$$(w_1 - v_1)^2 + (w_2 - v_2)^2 + (w_3 - v_3)^2 = v_1^2 + v_2^2 + v_3^2 + w_1^2 + w_2^2 + w_3^2 - 2|\vec{v}||\vec{w}|\cos$$

$$= (\cancel{w_1^2} - 2w_1v_1 + \cancel{v_1^2}) + (\cancel{w_2^2} - 2w_2v_2 + \cancel{v_2^2}) + (\cancel{w_3^2} - 2w_3v_3 + \cancel{v_3^2})$$

$$= \cancel{v_1^2} + \cancel{v_2^2} + \cancel{v_3^2} + \cancel{w_1^2} + \cancel{w_2^2} + \cancel{w_3^2} - 2|\vec{v}||\vec{w}|\cos\theta.$$

$$\Rightarrow \frac{\cancel{-2}[w_1v_1 + w_2v_2 + w_3v_3]}{\cancel{-2}} = \frac{\cancel{-2}|\vec{v}||\vec{w}|\cos\theta}{\cancel{-2}}$$

$$\cos\theta = \frac{w_1v_1 + w_2v_2 + w_3v_3}{|\vec{v}||\vec{w}|}$$

$$\Rightarrow \cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}$$

$90^\circ \quad \vec{v}\cdot\vec{w} = A$

$-$ $+$ $\cos\theta$

0

Fill in the blanks with right, acute, or obtuse.

If $\mathbf{v}\cdot\mathbf{w} > 0$, then θ is acute $\quad 0^\circ < \theta < 90^\circ$

If $\mathbf{v}\cdot\mathbf{w} = 0$, then θ is Right $\quad \theta = 90^\circ$

If $\mathbf{v}\cdot\mathbf{w} < 0$, then θ is obtuse $\quad 90^\circ < \theta < 180^\circ$

$$\theta = 90^\circ \quad \Rightarrow \cos 90^\circ = 0 = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\,|\vec{w}|}$$

Example 1: Find the angle between $\mathbf{a} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$.

$$\cos\theta = \frac{a \cdot b}{|a|\,|b|} = \frac{-21}{\sqrt{4^2+2^2+3^2}\;\sqrt{2^2+1^2+5^2}} \Rightarrow \theta = 135^\circ.$$

$$a \cdot b = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} = -8 + 2 - 15 = -21$$

Example 2

Find a if $\mathbf{p} = 3\mathbf{i} + 7\mathbf{j}$ and $\mathbf{q} = a\mathbf{i} - 4\mathbf{j}$ are perpendicular.

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} a \\ -4 \end{pmatrix} = 0$$

$$3a - 28 = 0$$

$$a = \frac{28}{3}$$