Poisson Distribution

The distribution of the number of events in a "random process".

Examples:

Random Process	**Event**
Telephone calls in a fixed time interval	**Number of wrong calls in an hour (Time dependent)**
Accident in a factory	**Number of Accident in a day (Time dependent)**
Flaws in a glass Panel	**Number of flaws per square cm (Area dependent)**
Bacteria in Milk	**Number of bacteria per 2 liter (Volume dependent)**

- **An event is as likely to occur in one given interval as it is in another.**
- **Events occur uniformly is proportional to the size of the time interval, area, or volume.**

The Poisson distribution formula:

$$P(X = x) = \frac{e^{-\mu}\mu^{x}}{x!} \textbf{ where } \mu = \lambda t \text{ and } x = 0, 1, 2, 3, \ldots\ldots$$

The Poisson Notation:

$X \sim Pn(\lambda t)$: The random variable x has a poisson distribution with parameter λt .

Where λ is a rate per unit and t is a time interval.

- X is the number of event in a time interval of length t with rate λ per unit time
-

Expected Value and Variance of the Poisson Distribution

$$E(x) = \mu \textbf{ and } Var(x) = \mu$$

$\not{f} \frac{1}{3}$ $\lambda = \mu = m$ (calc, Book, IB Booklet)

Ex) Faults occur on a piece of string at an average rate of one every three meters. Bobbins, each containing 5 meters of this string, are to be used. What is the probability that a randomly selected bobbin will contain.

a. Two faults.
b. At least two faults.

$m = \mu = \lambda = \left(\frac{1}{3}\right) \cdot 5 = \frac{5}{3}$

$P(X=2) = \boxed{\dfrac{\left(\frac{5}{3}\right)^2 e^{-5/3}}{2!}} = \text{poissonpdf}\left(\frac{5}{3}, 2\right)$

$1 - P(X=0) - P(X=1)$

$1 - \dfrac{\left(\frac{5}{3}\right)^0 \left(e^{-5/3}\right)}{0!} - \dfrac{\left(\frac{5}{3}\right)^1 e^{-5/3}}{1!}$

$1 - \text{poissoncdf}\left(\frac{5}{3}, 1\right)$

Ex) A radioactive source emits particles at an average rate of one every 12 seconds. Find the probability that at most 5 particles are emitted in one minute.

One minute = 60 sec.

$\mu = m = \left(\frac{1}{12}\right)(60) = 5.$

$P(X \leq 5)$

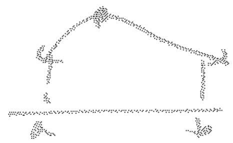
$= P(X=0) + P(X=1) \cdots P(X=5)$

$= \dfrac{(5)^0 (e^{-5})}{0!} + \dfrac{(5)^1 (e^{-5})}{1!} \cdots \dfrac{(5)^5 (e^{-5})}{5!} = \text{poissoncdf}(5, 5)$

Ex) A typist finds that they make two mistakes, on average, every three pages. Assuming that the number of errors per page follows a poisson distribution, what are the chances that there will be 2 mistakes in the nexst page they type?

Continuous Probability Density Fucntions

> A continuous probability density functions (PDF) is a function f(x) such that $f(x) \geq 0$ on a given interval $[a,b]$ and $\int_a^b f(x)dx = 1$.



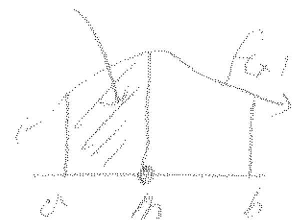
$\sum_{i=1}^{n} P(x_i) = 1$ $\frac{1}{2}$

The Properties:

1) The mode is the value of x at the maximum value of f(x) on $[a,b]$.

$f'(x) = 0$ Solve for x.

2) The median m is the solution for m of equation $\int_a^m f(x)dx = \frac{1}{2}$



3) The mean is defines as $\int_a^b xf(x)dx = \mu$

$\sum_{i=1}^{n} x_i P(x_i) = E(x) = \mu$ = mean

4) The variance is defined as $\int_a^b x^2 f(x)dx - \mu^2 = \text{var}(x)$

$\text{var}(x) = E(x^2) - (E(x))^2$

Ex) $f(x) = \begin{cases} -0.2x(x-b) & 0 \leq x \leq b \\ 0 & elsewhere \end{cases}$ is a probability density function.

a) Find b.

$\int_0^b -0.2x(x-b)dx = 1$

$\Rightarrow \int_0^b (-0.2x^2 + 0.2bx)dx = \left[\frac{-0.2}{3}x^3 + \frac{0.2bx^2}{2}\right]_{x=0}^{x=b}$

$= \frac{-0.2b^3}{3} + \frac{0.2b^3}{2} = 1$

$= \frac{-0.4b^3 + 0.6b^3}{6} = 1$

$\Rightarrow b = \sqrt[3]{30}$.

b) Find the mean

$\int_a^b x f(x)dx = \int_0^{\sqrt[3]{30}} x^2(-0.2(x-\sqrt[3]{30}))dx$

$= 1.554.. \approx \boxed{1.55}$

c) Find the variance and the standard deviation.

$\text{var}(x) = \int_0^{\sqrt[3]{30}} x^2\left[-0.2x(x-\sqrt[3]{30})\right]dx - (1.554..)^2$

≈ 0.482.

Practice) $f(x)=\begin{cases} kx^2(x-6) & 0 \le x \le 5 \\ 0 & elsewhere \end{cases}$ is a probability density function.

a) Find k. $\int_0^5 kx^2(x-6)\,dx = 1 \Rightarrow \int_0^5 k(x^3-6x^2)\,dx$

$\boxed{k = \frac{-4}{375}}$

b) Find the mode

$\frac{d}{dx}\left(\frac{-4}{375}x^2(x-6)\right)=0$ Solve for X

$\boxed{X = 4}$

c) Find the ~~mean~~ median.

$\frac{-4}{375}\int_0^m (x^3-6x^2)\,dx = \frac{1}{2}$

$\frac{-4}{375}\left[\frac{x^4}{4} - \frac{6x^2}{3}\right]_{x=0}^{x=m} = \frac{1}{2}$

$\therefore \frac{-4}{375}\left(\frac{m^4}{4} - 2m^3\right] = \frac{1}{2} \cdot 2$

$\frac{-8}{375}\left[\frac{m^4}{4} - 2m^3\right] = 1$

$-2m^4 + 16m^3 = 375$

$-2m^4 + 16m^3 - 375 = 0$

Use the polynomial Root finder.

$\boxed{m = 3.46}$

d) Find the variance and the standard deviation.

The Normal Distribution (Distribution for a continuous random variable)

Manu naturally occurring phenomena have a distribution that is normal or close to normal (a Bell-shaped distribution).

The Standard Normal Curve

$$f(z)=\frac{1}{\sqrt{2\pi}}e^{-(z^2/2)} \textbf{ (for } -\infty < z < \infty\textbf{) and where } z=\frac{x-\mu}{\sigma} \text{ (Standardized variable)}$$