1) Derive the Poisson distribution $p(x)=\frac{e^{-\mu}\mu^{x}}{x!}$ from the Binomial distribution $p(x)={}_{n}C_{x}p^{x}(1-p)^{n-x}$ as $n\to\infty$.

Binomial : $P={}_{n}C_{x}\,p^{x}(1-p)^{n-x}$ $\qquad E(x)=\mu=n\cdot p$, $\quad p=\frac{\mu}{n}$

$$=\frac{n!}{(n-x)!\,x!}\left(\frac{\mu}{n}\right)^{x}\left(1-\frac{\mu}{n}\right)^{n-x}$$

$$=\frac{n!}{(n-x)!\,x!}\,\frac{\mu^{x}}{n^{x}}\boxed{\left(1-\frac{\mu}{n}\right)^{n}}\left(\frac{n}{n}-\frac{\mu}{n}\right)^{-x}$$

+5

$$=\frac{n!}{(n-x)!\,x!}\,\frac{\mu^{x}}{n^{x}}\left(\frac{n}{n-\mu}\right)^{x}\left(1-\frac{\mu}{n}\right)^{n}$$

$$=\frac{n!}{(n-x)!\,x!}\left(\frac{\mu^{x}}{\cancel{n^{x}}}\right)\left(\frac{\cancel{n^{x}}}{(n-\mu)^{x}}\right)\left(1-\frac{\mu}{n}\right)^{n}.$$

proof continue Next page.

2) Derive $E(x)=\mu$ for the Poisson Distribution.

+3

$x=y+1\Rightarrow y=x-1$

$x=1\Rightarrow y=0$

$x\to\infty\Rightarrow y\to\infty$

Binomial : $E(x)=n\cdot p$

poisson: $\sum x\cdot p=\sum_{x=1}^{\infty}x\cdot\frac{e^{-\mu}\cdot\mu^{x}}{x!}$

$$=\sum_{y=0}^{\infty}\frac{\cancel{(y+1)}e^{-\mu}\mu^{y+1}}{\cancel{(y+1)}!\,y!}=\sum_{y=0}^{\infty}\frac{e^{-\mu}\cdot\mu\cdot\mu^{y}}{y!}$$

$$=e^{-\mu}\cdot\mu\cdot\sum_{y=0}^{\infty}\frac{\mu^{y}}{y!}=\cancel{e^{-\mu}}\cdot\mu\,\cancel{e^{\mu}}=\boxed{\mu}$$

Notes: $\sum_{y=0}^{\infty}\frac{\mu^{y}}{y!}=1+\frac{\mu}{1}+\frac{\mu^{2}}{2!}+\frac{\mu^{3}}{3!}\cdots$

$=e^{\mu}$ by M Series.

proof for $\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{n} = e^{-\mu}$

$$\lim_{n\to\infty} \ln\left(1-\frac{\mu}{n}\right)^{n} = \ln L$$

$$= \lim_{n\to\infty} \frac{\left(\ln\left(1-\frac{\mu}{n}\right)\right)'}{\left(\frac{1}{n}\right)'} \left(\frac{0}{0}\right) = \ln L$$

+2

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{1-\frac{\mu}{n}}\right)\left(\frac{\mu}{\cancel{n^2}}\right)}{\frac{-1}{\cancel{n^2}}} = -\mu = \ln L$$

$$\boxed{L = e^{-\mu}}$$

poisson $n\to\infty$

$$\Rightarrow \lim_{n\to\infty} \frac{n!}{(n-x)!\,x!} \frac{\mu^x}{(n-\mu)^x} \boxed{\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{n}}$$

$e^{-\mu}$ +5

$$= \lim_{n\to\infty} \frac{(n)(n-1)(n-2)\cdots(n-(x-1))\cancel{(n-x)!}}{\cancel{(n-x)!}\,\boxed{x!}\,(n-\mu)^x} \boxed{\mu^x}\boxed{e^{-\mu}}$$

$$= \lim_{n\to\infty} \frac{\overbrace{n(n-1)\cdots(n-x+1)}^{n^x+\cdots}}{\underbrace{(n-\mu)^x}_{n^x+\cdots}} \cdot \frac{\mu^x}{x!} e^{-\mu}$$

$$= \frac{\overset{1}{\cancel{}}\,\mu^x \cdot e^{-\mu}}{x!}$$