IB Math 3: Ratio Test WS #2 **Name:** ____________________ **Period:** __________

1. Let T= $\sum_{k=1}^{\infty} \frac{k}{2^k}$, where each term $a_k = \frac{k}{2^k}$

a) Consider the tests we have for convergence at this point. Which, if any, can be used to determine if T converges?

Not sure what test will work.

b) Evaluate $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ and call the limit r.

$a_{k+1} = \frac{k+1}{2^{k+1}}$, $a_k = \frac{k}{2^k}$.

$\lim_{k\to\infty}\left(\frac{k+1}{2^{k+1}}\right)\left(\frac{2^k}{k}\right) = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)\left(\frac{\cancel{2^k}}{\cancel{2^k}\cdot 2}\right) = \lim_{k\to\infty} \frac{k+1 \div k}{2k \div k} =$

c) Since $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = r$, that means that eventually, $a_{k+1} \approx r a_k$. Therefore, T resembles what type of series?

$r = \frac{1}{2} \Rightarrow a_{k+1} \approx r \cdot a_k \Rightarrow$ G Series.

$\lim_{k\to\infty} a_{k+1} \approx \lim_{k\to\infty} r \cdot a_k$.

d) Given the value of r, would you conclude that T converges or diverges?

$|r| < 1 \Rightarrow$ G converges

$|r| > 1 \Rightarrow$ G diverges.

e) What would you have concluded if r had been greater than 1?

The Ratio Test

- **Given the series $\sum a_k$ with $a_k > 0$, suppose that $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L$, Then**
  - **If L< 1, then $\sum a_k$ converges**
  - **If L > 1 or if is infinite, then $\sum a_k$ diverges**
- **If L = 1, the test is inconclusive and another test must be tried.**

Example) Test the series, $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ for convergence.

By Ratio test.

$a_k = \frac{3^k}{k!}$ $\qquad a_{k+1} = \frac{3^{k+1}}{(k+1)!}$

$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\left(\frac{3^{k+1}}{(k+1)!}\right)\left(\frac{k!}{3^k}\right) = \lim_{k\to\infty}\left(\frac{k!}{(k+1)!}\right)\left(\frac{\cancel{3^k}\cdot 3}{\cancel{3^k}}\right)$

$= \lim_{k\to\infty} \frac{\cancel{k!}}{\cancel{k!}(k+1)} \cdot 3 = \lim_{k\to\infty} \frac{3}{k+1} = 0 < 1$

$\therefore \sum \frac{3^n}{n!}$ converges

2. Test the following infinite series for convergence. Name the test you use. Work on graph paper.

a) $\sum_{k=1}^{\infty} \frac{k!}{k^3}$

By Ratio test.

$$\lim_{k\to\infty} \frac{(k+1)!}{(k+1)^3} \cdot \frac{k^3}{k!}$$

$$= \lim_{k\to\infty} \frac{(k!)(k+1)}{k!} \cdot \left(\frac{k}{k+1}\right)^3$$

$$= \lim_{k\to\infty} \frac{(k+1)(k^3)}{(k^3+3k^2+3k+1)} = \infty > 1$$

$\therefore \sum \frac{k!}{k^3}$ diverges.

b) $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

By Limit Comparison.

$$\sum \frac{k}{k^2+1} < \sum \frac{k}{k^2} = \sum \frac{1}{k}.$$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \left(\frac{k}{k^2+1}\right)\left(\frac{k}{1}\right) = \lim_{k\to\infty} \frac{k^2}{k^2+1} = 1$$

$$\sum \frac{1}{k} \Rightarrow \text{Int. test} \lim_{a\to\infty} \int_1^a \frac{1}{k}\,dk = \lim_{a\to\infty} \ln a - \ln 1$$

$= \infty$. diverges.

$\therefore \sum \frac{k}{k^2+1}$ diverges.

c) $\sum_{k=1}^{\infty} \frac{1}{2^k+1}$

By Direct Comparison

$$\sum \frac{1}{2^k+1} < \sum \frac{1}{2^k}.$$

$\sum \frac{1}{2^k}$ converges by G series $|r| = |\frac{1}{2}| < 1$.

$\therefore \sum \frac{1}{2^k+1}$ converges.

d) $\sum_{k=11}^{\infty} \frac{\ln k}{e^k}$

By Ratio test.

$$\lim_{k\to\infty} \left(\frac{\ln k+1}{e^{k+1}}\right)\left(\frac{e^k}{\ln k}\right)$$

$$= \lim_{k\to\infty} \left(\frac{\ln k+1}{\ln k}\right)\left(\frac{e^k}{e^k \cdot e}\right)$$

$$= \frac{1}{e} < 1.$$

$\therefore \sum \frac{\ln k}{e^k}$ converges

e) $\sum_{k=11}^{\infty} \frac{2^k}{3^k}$

By G Series test.

$$|r| = \left|\frac{2}{3}\right| < 1$$

$\therefore \sum \frac{2^k}{3^k}$ converges

f) $\sum_{k=11}^{\infty} \frac{3^k}{k^2}$

By Ratio test.

$$\lim_{k\to\infty} \frac{3^{k+1}}{(k+1)^2} \cdot \frac{k^2}{3^k}$$

$$= \lim_{k\to\infty} \frac{3^k \cdot 3}{3^k} \cdot \frac{k^2}{k^2+2k+1}$$

$$= \lim_{k\to\infty} \frac{3k^2}{k^2+2k+1} = 3 > 1$$

$\therefore \sum \frac{3^k}{k^2}$ diverges.