IB Math HL1 Intro to Related Rates

Related Rates problems always deal with how a quantity is changing over **time**.

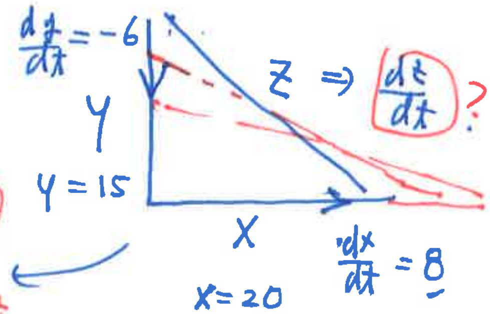
We will solve the word problems using these steps:

- Step 1: Draw a large, clear **diagram** of the situation.
- Step 2: Write down the information, label the diagram, and make a distinction between **variables** and **constants**.
- Step 3: Write an **equation** connecting the variables.
- Step 4: **Differentiate** the equation with respect to t to obtain a differential equation.
- Step 5: Solve for the **particular case** which is at some instant in time.

Before do Step 1-Step 5, practice for Step 4 sand 5

Example)

Equation	Find this	When
$x^2+y^2=z^2$	$\frac{dz}{dt}$	$x=20$ cm, $y=15$ cm, if $\frac{dx}{dt}=8$ cm/min, $\frac{dy}{dt}=-6$ cm/min and z > 0



Step 4: Differentiate $x^2+y^2=z^2$ with respect to t.

$$\frac{d}{dx}(x^2)\cdot\frac{dx}{dt}+\frac{d}{dy}(y^2)\frac{dy}{dt}=\frac{d}{dz}(z^2)\cdot\frac{dz}{dt}$$

Step 5: Solve for the particular case.

$$\Rightarrow 2x\cdot\frac{dx}{dt}+2y\cdot\frac{dy}{dt}=2z\frac{dz}{dt}$$

$$\cancel{(2)}(20)(8)+\cancel{(2)}(15)(-6)=(2)(z)\cdot\frac{dz}{dt}$$

$$=\cancel{(2)}(25)\frac{dz}{dt}$$

$$z=\sqrt{20^2+15^2}=25.$$

$$\frac{dz}{dt}=2.8.\ \left(\frac{14}{5}\right)\frac{cm}{min}$$

Practice

	Equation	Find this	When
1.	$y=\sqrt{x}$	$\frac{dy}{dt}=?$ $\frac{dy}{dt}=\frac{1}{2\sqrt{4}}\cdot 3=\frac{3}{4}\frac{ft}{sec}$ $y=\sqrt{x}\Rightarrow(1)\cdot\frac{dy}{dt}=\frac{1}{2\sqrt{x}}\cdot\frac{dx}{dt}.$	$x=4$ feet and $\frac{dx}{dt}=3$ feet/sec
2.	$y=2(x^2-3x)$ $=2x^2-6x$	$\frac{dx}{dt}\Rightarrow\frac{2}{[(4)(25)-6]}=\frac{1}{47}\approx 0.0213\ \frac{ml}{hr}$ $\frac{dy}{dt}=2(2x-3)\frac{dx}{dt}=(4x-6)\cdot\frac{dx}{dt}.$	$x=25$ mL and $\frac{dy}{dt}=2$ mL/hour
3.	$xy=4$ $y=\frac{4}{x}.$	$\frac{dy}{dt}=\left(\frac{-4}{8^2}\right)(10)=\frac{-5}{8}\frac{gal}{day}$ $\frac{dy}{dt}=\frac{-4}{x^2}\cdot\frac{dx}{dt}.$	$x=8$ gallons and $\frac{dx}{dt}=10$ gallons/day
4.	$x^2-y^2=25$	$\frac{dx}{dt}=\frac{12\cdot 4}{13}=\frac{48}{13}\frac{cm}{min}$ $2x\cdot\frac{dx}{dt}-2y\cdot\frac{dy}{dt}=0$ $\frac{dx}{dt}=\frac{y\cdot\frac{dy}{dt}}{x}$	$x=13$ cm, $\frac{dy}{dt}=4$ cm/min and y > 0 $y=\sqrt{x^2-25}=\sqrt{13^2-25}=12.$
5.	$V=\frac{1}{3}\pi r^2h$	$\frac{dh}{dt}=1\ \frac{in}{min}$	$r=9$ in and $h=4$ in, $\frac{dV}{dt}=99\pi$ in^3/min and $\frac{dr}{dt}=3$ in/min

$$\frac{dV}{dt}=\frac{1}{3}\pi\left(2r\cdot h\cdot\frac{dr}{dt}+r^2\cdot\frac{dh}{dt}\right)$$

$$99\pi=\frac{1}{3}\pi\left(2\cdot(9)(4)(3)+(9)^2\cdot\frac{dh}{dt}\right)$$

IB Math HL1 Related Rates Day one Examples

Reminder: Related Rates Problem-Solving Steps

Step 1: Draw a large, clear **diagram** of the situation.

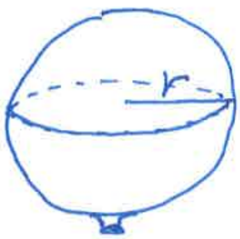
Step 2: Write down the information, label the diagram, and make a distinction between **variables** and **constants.**

Step 3: Write an **equation** connecting the variables.

Step 4: **Differentiate** the equation with respect to t to obtain a differential equation.

Step 5: Solve for the **particular case** which is at some instant in time.

1. A spherical balloon is inflating at a rate of 27π in³/sec. How fast is the radius of the balloon increasing when the radius is 3 in?



$\frac{dV}{dt} = 27\pi \frac{in^3}{sec.}$

$\frac{dr}{dt} = ?$

$r = 3$ in.

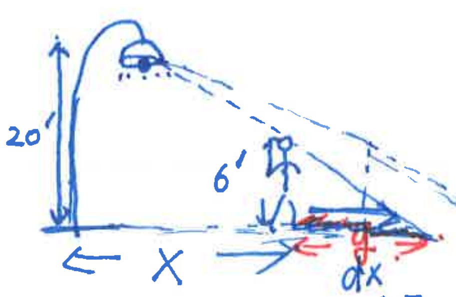
$\Rightarrow V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\cdot\frac{dr}{dt}$

$27\pi = \frac{4}{3}\pi(3\cdot 3^2)\boxed{\frac{dr}{dt}}$

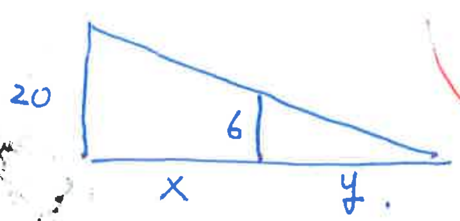
$\frac{dr}{dt} = \frac{3}{4}\ \frac{in}{sec}$

2. A person 6 feet tall walks away from a streetlight 20 feet high at the rate of 7 ft/sec. At what rate is the length of the person's shadow changing?



$\frac{dy}{dt} = ?$

$\frac{dx}{dt} = \frac{7\,ft}{sec.}$



$\frac{20}{x+y} = \frac{6}{y}$

$20y = 6x + 6y$

$14y = 6x$

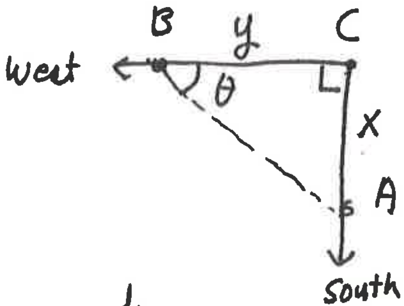
$y = \frac{3}{7}x$

$\frac{dy}{dt} = \frac{3}{7}\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{3}{\cancel{7}}(\cancel{7})$

$= 3\ \frac{ft}{sec}$

3. Cars A and B leave a town C at the same time. Car A heads due south at a rate of 50 km/hr, and car B heads due west at a rate of 60 km/hr. At what rate is angle $C\hat{B}A$ changing after 3 hours?



$$\frac{dx}{dt} = 50\frac{km}{hr}.$$

$$\frac{dy}{dt} = 60\frac{km}{hr}.$$

$$\frac{d\theta}{dt} = ?$$

$t = 3hr.$

$x = 50 \cdot 3 = 150km$

$y = 60 \cdot 3 = 180km.$

$\theta = \tan^{-1}\left(\frac{150}{180}\right) \approx .695 \text{ rad}.$

$$\tan\theta = \frac{x}{y}.$$

$$\sec^2\theta \cdot \boxed{\frac{d\theta}{dt}} = \frac{(1)\cdot y \cdot \frac{dx}{dt} - (x)(1)\cdot\frac{dy}{dt}}{y^2}.$$

$$\sec^2(.695)\cdot\frac{d\theta}{dt} = \frac{(1)(180)(50)-(1)(150)(60)}{(180)^2}$$

$$\frac{d\theta}{dt} = 0\ \frac{rad}{hr}$$

Related Rates Day two Examples)

4. A hot-air balloon rising straight up from a field is tracked by a range finder 500 feet from the lift-off point. At the moment the range-finder's angle of elevation is $\frac{\pi}{4}$, the angle is increasing at a rate of 0.14 radians per minute. How fast is the balloon rising at that moment?

$$\theta = \frac{\pi}{4}\text{ rad}.$$

$$\frac{d\theta}{dt} = 0.140\frac{rad}{min}.$$

$$\frac{dh}{dt} = ?$$

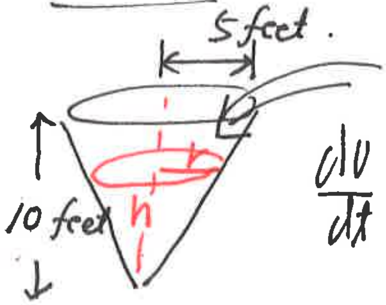
$$\tan\theta = \frac{h}{500}.$$

$$\sec^2\theta\cdot\frac{d\theta}{dt} = \frac{1}{500}\boxed{\frac{dh}{dt}}$$

$$\sec^2\left(\frac{\pi}{4}\right)(0.140)(500) = \frac{dh}{dt}.$$

$$\frac{dh}{dt} = 140\frac{ft}{min}$$

5. Water runs into a conical tank, standing point down, at the rate of 9 ft^3/min. The tank's height is 10 feet and its base radius is 5 feet. How fast is the water level rising when the water is 6 feet deep?



$$\frac{dV}{dt} = 9\,\frac{ft^3}{min}$$

$$\frac{dh}{dt} = ?$$

$h = 6$ feet.

$$\frac{1}{2} = \frac{5}{10} = \frac{r}{h}$$

$$r = \frac{1}{2}h$$

$$r = \left(\frac{1}{2}\right)(6) = 3\text{ feet}$$

$$V = \frac{1}{3}\pi r^2 h.$$

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 \cdot h.$$

$$= \frac{1}{3}\pi \cdot \frac{1}{4}h^3$$

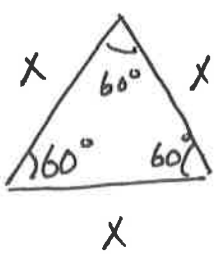
$$V = \frac{\pi}{12}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{\cancel{12}_{4}} \cdot \cancel{3} \cdot h^2 \cdot \frac{dh}{dt}$$

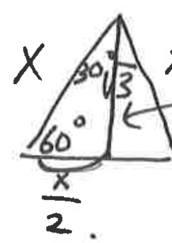
$$9 = \left(\frac{\pi}{4}\right)(6)^2 \cdot \boxed{\frac{dh}{dt}} \Rightarrow \frac{dh}{dt} = \frac{1}{\pi}\,\frac{ft}{min}$$

$$\approx .318\,\frac{ft}{min}$$

6. The sides of an equilateral triangle are increasing at the rate of 27 in/sec. How fast is the triangle's area increasing when the sides of the triangle are each 18 in. long?



$$A = \frac{\sqrt{3}}{4}x^2$$



$$h = \frac{\sqrt{3}}{2}x$$

$$A = \frac{1}{2} \cdot x \cdot h$$

$$= \left(\frac{1}{2}\right)x\left(\frac{\sqrt{3}}{2}x\right)$$

$$= \frac{\sqrt{3}}{4}x^2.$$

$$\frac{dx}{dt} = 27\,\frac{in}{sec.}$$

$$\frac{dA}{dt} = ?$$

$x = 18$ in.

$$\frac{dA}{dt} = \left(\frac{\sqrt{3}}{4}\right)(2x)\frac{dx}{dt}.$$

$$\frac{dA}{dt} = \left(\frac{\sqrt{3}}{\cancel{4}_{2}}\right)(\cancel{2})(\cancel{18}^{9})(27)$$

$$= 243\sqrt{3}\,\frac{in^2}{sec}.$$