IB Math HL2 Indeterminate Limit, L'Hopital's rule ,and Improper Integral (review)

l'Hopital's rule:

Theorem: Let $f(x)$ and $g(x)$ be the functions that are differentiable on an open interval (a,b) containing c. Assum that $g'(x) \neq 0$ for all x in (a,b). If the limit of $\frac{f(x)}{g(x)}$ as x approaches c produces the indeterminate form $\lim \frac{0}{0}$, $\lim \frac{\infty}{\infty}$, $\lim \frac{-\infty}{-\infty}$, $\lim \frac{-\infty}{\infty}$, $\lim \frac{\infty}{-\infty}$ and $\lim 0 \cdot \infty$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Example 1) $\lim_{x \to 0} \frac{e^{2x}-1}{x}$ $\left(\frac{e^0-1}{0} = \frac{1-1}{0} = \frac{0}{0}\right)$

$$\lim_{x \to 0} \frac{(e^{2x}-1)'}{(x)'} = \lim_{x \to 0} \frac{2 \cdot e^{2x}}{1} = \frac{2 \cdot e^0}{1} = \boxed{2}$$

Example 2) $\lim_{x \to \infty} e^{-x}\sqrt{x} = \lim_{x \to \infty} \frac{\sqrt{x}}{e^x}$ $\left(\frac{\infty}{\infty}\right)$

$$= \lim_{x \to \infty} \frac{(\sqrt{x})'}{(e^x)'} = \lim_{x \to \infty} \frac{\frac{1}{2} \cdot x^{-\frac{1}{2}}}{e^x} = \lim_{x \to \infty} \frac{1}{2\sqrt{x} \cdot e^x} = \frac{1}{\infty} = \boxed{0}$$

Example 4) $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$ (Use of substitution)

$\left(\begin{array}{l} x \to \infty \\ u \to 0 \end{array}\right)$ $\left(x = \frac{1}{u} \Rightarrow u = \frac{1}{x}\right)$

$$\lim_{u \to 0} \frac{1}{u} \cdot \sin u = \lim_{u \to 0} \frac{\sin u}{u} \left(\frac{0}{0}\right) = \lim_{u \to 0} \frac{(\sin u)'}{u'} = \lim_{u \to 0} \frac{\cos u}{1} = \frac{1}{1} = \boxed{1}$$

Example 5) $\lim_{x \to 0^+} (\sin x)^x$ (Indeterminate form 0^0)

(0^0)

$$\lim_{x \to 0^+} \ln(\sin x)^x = \ln L$$

$$= \lim_{x \to 0^+} x \ln \sin x = \lim_{x \to 0^+} \frac{\ln \sin x}{\frac{1}{x}} \left(\frac{-\infty}{\infty}\right) = \lim_{x \to 0^+} \frac{(\ln \sin x)'}{\left(\frac{1}{x}\right)'}$$

$$= \lim_{x \to 0^+} \frac{\left(\frac{1}{\sin x}\right)(\cos x)}{-\frac{1}{x^2}}$$

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$$\lim_{x\to 0^+} \frac{(\cot x)'}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-\csc^2 x}{2x^{-3}}$$

$$= \lim_{x\to 0} -\frac{1}{2} x^3 \cdot \csc^2 x \, . = \lim_{x\to 0} -\frac{x^3}{2\sin^2 x} \, . \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-3x^2}{4\sin x \cos x} \, . = \lim_{x\to 0} \left(-\frac{3}{4}\right)\left(\cancel{\frac{x}{\sin x}}^{\,1}\right)\left(\frac{x}{\cos x}\right)$$

$$= \left(-\frac{3}{4}\right)(1)(0) = 0$$

$$\ln L = 0 \Rightarrow \boxed{L = e^0 = 1}$$

Example 6) $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$ (Indeterminate form $\infty-\infty$)

$$= \lim_{x\to 1^+}\left(\frac{(x-1)-\ln x}{(\ln x)(x-1)}\right) \quad \left(\frac{1-1-\ln 1}{(\ln 1)(1-1)} = \frac{0}{0}\right)$$

$$= \lim_{x\to 1^+} \frac{\left[1-\frac{1}{x}\right]x}{\left[\left(\frac{1}{x}\right)(x-1)+(\ln x)(1)\right]x}$$

$$= \lim_{x\to 1^+} \frac{(x-1)}{(x-1)+x\ln x} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 1^+} \frac{(1)}{1+\ln x+\cancel{x}\cdot\frac{1}{\cancel{x}}} \quad 1$$

$$= \lim_{x\to 1^+} \frac{1}{1+\ln x+1} = \boxed{\frac{1}{2}}$$

Example 7) $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$ (Indeterminate form 1^∞)

$$\ln \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \ln L$$

$$\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^{x} = \ln L$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty} \frac{\left(\frac{1}{1+\frac{1}{x}}\right)\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = 1$$

$$1 = \ln L$$

$$\boxed{L = e}$$

Improper Integral :

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE INTEGRATION LIMITS

1. If f is continuous on the interval $[a, \infty)$, then
$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$
2. If f is continuous on the interval $(-\infty, b]$, then
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$
3. If f is continuous on the interval $(-\infty, \infty)$, then
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$
where c is any real number (see Exercise 120).

In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Example 1) Evaluate $\int_0^{\infty} e^{-x}dx = \lim_{a\to\infty}\int_0^a e^{-x}dx = \lim_{a\to\infty}\left[-e^{-x}\right]_0^a$

$= \lim_{a\to\infty}\left[\frac{-1}{e^x}\right]_0^a = \lim_{a\to\infty}\left[\frac{-1}{e^a}+\frac{1}{1}\right] = \cancel{\frac{-1}{\infty}}+1 = \boxed{1}$ (converges to 1)

Example 2) Evaluate $\int_1^{\infty}(1-x)e^{-x}dx = \lim_{a\to\infty}\int_1^a\left(\frac{1-x}{e^x}\right)dx$

u	dv
$1-x$	$\oplus\ e^{-x}$
-1	$-\ {-e^{-x}}$
0	$\ominus\ e^{-x}$

$= \lim_{a\to\infty}\left[e^{-x}(x-\cancel{1})+\cancel{e^{-x}}\right]_1^a = \lim_{a\to\infty}\left[xe^{-x}\right]_1^a$

$= x\cdot e^{-x}$

$= \lim_{a\to\infty}\left[\frac{a}{e^a}\left(\frac{\infty}{\infty}\right) - \frac{1}{e}\right] = \lim_{a\to\infty}\cancel{\frac{1}{e^a}}^{0} - \frac{1}{e} = \boxed{-\frac{1}{e}}$ (converges to $-\frac{1}{e}$)

Example 3) Evaluate $\int_0^{\infty}\frac{1}{\sqrt{x}(x+1)}dx$

$\Rightarrow \lim_{a\to\infty}\int_0^a \frac{1}{\sqrt{x}((\sqrt{x})^2+1)}dx$.

$\left(\begin{array}{ll} u=\sqrt{x} & x\to\infty \\ du=\frac{1}{2}\cdot\frac{1}{\sqrt{x}} & u\to\infty. \end{array}\right)$

$= \lim_{a\to\infty}\int_0^a \frac{2}{(u^2+1)}du$

$= \lim_{a\to\infty}\left[2\arctan u\right]_0^a = \lim_{a\to\infty}\left[2\arctan a - \cancel{2\arctan 0}^{0}\right]$

$= 2\cdot\frac{\pi}{2} = \pi$ (converges to π)