Rolle's theorem & MVT HW Answers.

Hence, $\lim\limits_{x\to 0^+} x\sin x = \lim\limits_{x\to 0^+} e^{\frac{\ln x}{[\sin x]^{-1}}}$

$= e^0$

$= 1$

10 $x^{\frac{1}{x}} = (e^{\ln x})^{\frac{1}{x}} = e^{\frac{\ln x}{x}}$

Consider $\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$.

As $x \to \infty$, $\ln x \to \infty$

The limit has type $\frac{\infty}{\infty}$, so we can use l'Hôpital's Rule.

$\therefore \lim\limits_{x\to\infty} \dfrac{\ln x}{x} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{1} = 0$

$\therefore \lim\limits_{x\to\infty} e^{\frac{\ln x}{x}} = e^0 = 1$

$\therefore \lim\limits_{x\to\infty} x^{\frac{1}{x}} = 1$

EXERCISE F

1 **a** $f(x) = 3x^3 + 5x^2 - 43x + 35$ on $[-5, 2\frac{1}{3}]$

$f(-5) = f(2\frac{1}{3}) = 0$

$f'(x) = 9x^2 + 10x - 43$ for all $x \in \mathbb{R}$

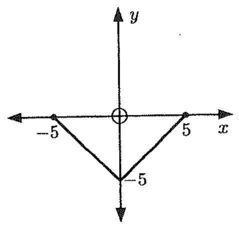
$\therefore$ $f(x)$ is continuous and differentiable for all $x \in [-5, 2\frac{1}{3}]$ and $f(-5) = f(2\frac{1}{3}) = 0$

$\Rightarrow$ Rolle's Theorem applies

$f'(c) = 0 \Leftrightarrow c = \dfrac{-5 \pm 2\sqrt{103}}{9}$

both of which lie in $]-5, 2\frac{1}{3}[$

b $f(x) = |x| - 5$ on $[-5, 5]$.



$f(x)$ is not differentiable at $x = 0$.

$\begin{cases} \text{If } x \geqslant 0, & f'_+(x) = 1 \\ \text{If } x < 0, & f'_-(x) = -1 \end{cases}$

Thus $f(x)$ is not differentiable for all $x \in [-5, 5]$

$\therefore$ Rolle's Theorem does not apply.

c $f(x) = 2 - \dfrac{1}{x+1}$ on $[-\frac{1}{2}, 7]$.

$f(x)$ is continuous for all $x \in \mathbb{R}$, $x \neq -1$

$\therefore$ is continuous on $[-\frac{1}{2}, 7]$

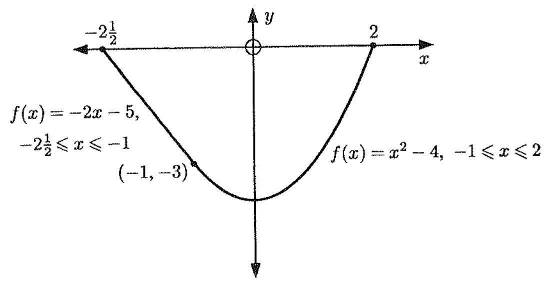
$f'(x) = \dfrac{1}{(x+1)^2}$ exists for all $x \in \mathbb{R}$, $x \neq -1$

$\therefore$ is differentiable on $[-\frac{1}{2}, 7]$

$f(-\frac{1}{2}) = 0$ and $f(7) = 1\frac{7}{8} \neq 0$

$\therefore$ Rolle's Theorem does not apply.

d $f(x) = \begin{cases} -2x - 5, & x < -1 \\ x^2 - 4, & x \geqslant -1 \end{cases}$ on $[-2\frac{1}{2}, 2]$



$f'(x) = \begin{cases} -2, & x < -1 \\ 2x, & x \geqslant -1 \end{cases}$

$\therefore f'_-(-1) = f'_+(-1) = -2$

$\therefore f'(-1) = -2$

$f(-1\frac{1}{2}) = f(2) = 0$ and $f(x)$ is continuous and differentiable on $[-2\frac{1}{2}, 2]$.

$\therefore$ Rolle's Theorem applies and $f'(c) = 0 \Leftrightarrow c = 0$ which lies in $]-2\frac{1}{2}, 2[$.

2 **a** **i** $f(x) = (x-1)(x-2)(x-4)(x-5)$ has zeros: 1, 2, 4, and 5

As $f(x)$ is a real polynomial, continuous and differentiable for all $x \in \mathbb{R}$, by Rolle's Theorem, there exist zeros of $f'(x)$ in the intervals $]1, 2[$ and $]2, 4[$ and $]4, 5[$.

So at least 3 real zeros exist for $f'(x)$.

ii $f(x)$ has sign diagram and graph:

We see that there are 3 turning points

$\therefore$ there are exactly 3 real distinct zeros of $f'(x)$.

b **i** $f(x) = (x-1)^2(x^2-9)(x-2)$

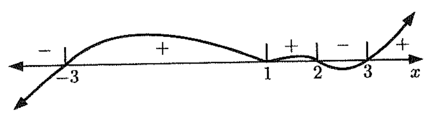
$= (x+3)(x-1)^2(x-2)(x-3)$

which has zeros: -3, 1, 2, and 3.

As $f(x)$ is a polynomial, continuous and differentiable for all $x \in \mathbb{R}$, by Rolle's Theorem, there exist zeros of $f'(x)$ in the intervals $]-3, 1[$ and $]1, 2[$ and $]2, 3[$.

So at least 3 zeros exist for $f'(x)$.

ii $f(x)$ has sign diagram and graph:



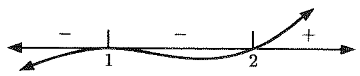
As $y = f(x)$ has 4 turning points, $f'(x)$ has 4 distinct real zeros, 3 guaranteed by Rolle's Theorem and $x = 1$.

c **i** $f(x) = (x-1)^2(x^2+9)(x-2)$ has two real zeros: 1 and 2.

As $f(x)$ is a polynomial, continuous and differentiable for all $x \in \mathbb{R}$, by Rolle's Theorem, there exists a zero of $f(x)$ in $]1, 2[$.

So at least one zero exists for $f'(x)$.

ii $f(x)$ has sign diagram and graph:



$\therefore$ $f'(x)$ has exactly 2 real distinct zeros, the one guaranteed by Rolle's Theorem and the one at $x = 1$.

3 a $f(x) = x^3$ is continuous on $[-2, 2]$ and differentiable on $]-2, 2[$.

Thus, by the MVT, there exists c in $[-2, 2]$ such that

$f(2) - f(-2) = f'(c)(2 - -2)$

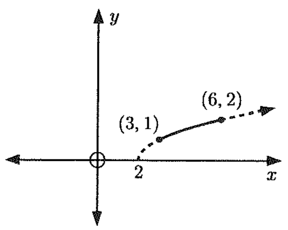
$\Rightarrow$ $8 - (-8) = f'(c) \times 4$

$\Rightarrow$ $f'(c) = 4$

As $f'(x) = 3x^2$, $f'(c) = 4 \Leftrightarrow 3c^2 = 4$

$\Leftrightarrow c = \pm\frac{2}{\sqrt{3}}$

b



$f(x) = \sqrt{x-2}$ is continuous on $[3, 6]$ and differentiable on $]3, 6[$.

$\therefore$ by the MVT, there exists c on $[3, 6]$ such that

$f(6) - f(3) = f'(c)(6-3)$

$\therefore$ $2 - 1 = f'(c) \times 3$

$\therefore$ $f'(c) = \frac{1}{3}$

Now $f'(x) = \dfrac{1}{2\sqrt{x-2}}$

$\therefore$ $\dfrac{1}{2\sqrt{c-2}} = \frac{1}{3}$

$\therefore$ $\sqrt{c-2} = \frac{3}{2}$

$\therefore$ $c - 2 = \frac{9}{4}$

$\therefore$ $c = 4\frac{1}{4}$

c $f(x) = x + \dfrac{1}{x}$ has $f'(x) = 1 - \dfrac{1}{x^2}$

$\therefore$ $f(x)$ is continuous and differentiable for all $x \in \mathbb{R}$, $x \neq 0$

$\therefore$ $f(x)$ is continuous on $[1, 3]$ and differentiable on $]1, 3[$ {as $0 \notin$ these intervals}

$\therefore$ by the MVT, there exists $c \in [1, 3]$ such that

$f(3) - f(1) = f'(c)(3-1)$

$\Rightarrow$ $3\frac{1}{3} - 2 = 2f'(c)$

$\Rightarrow$ $f'(c) = \frac{2}{3}$

and $1 - \dfrac{1}{c^2} = \frac{2}{3} \Leftrightarrow \dfrac{1}{c^2} = \frac{1}{3}$

$\Leftrightarrow c = \pm\sqrt{3}$

$\Leftrightarrow c = \sqrt{3}$, as $c > 0$

4 a $f(x) = \sqrt{x}$ $\therefore$ $f'(x) = \dfrac{1}{2\sqrt{x}}$

$f(x)$ is continuous for all $x \geqslant 0$ and $f'(x)$ exists for all $x > 0$

Thus, on $[49, 51]$ $f(x)$ is continuous and differentiable.

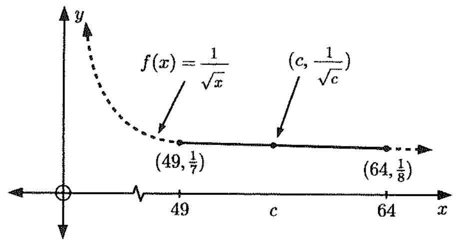
Hence, by the MVT, there exists $c \in [49, 51]$ such that

$\sqrt{51} - \sqrt{49} = f'(c)(51 - 49)$

$\Rightarrow$ $\sqrt{51} - 7 = 2 \times \dfrac{1}{2\sqrt{c}}$

$\Rightarrow$ $\sqrt{51} - 7 = \dfrac{1}{\sqrt{c}}$

b

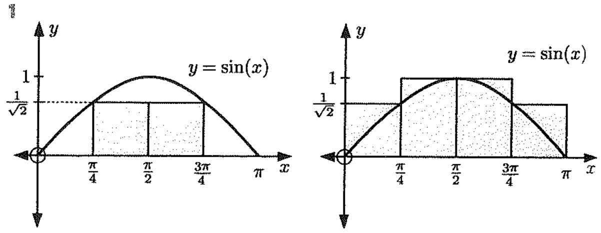


c From the graph $\frac{1}{8} < \dfrac{1}{\sqrt{c}} < \frac{1}{7}$

$\therefore$ $\frac{1}{8} < \sqrt{51} - 7 < \frac{1}{7}$

EXERCISE G.1

1



$L_4 = \frac{1}{\sqrt{2}} \times \frac{\pi}{2}$

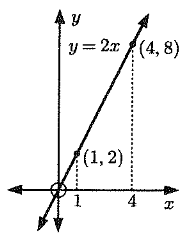
$= \frac{\pi}{2\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$

$= \frac{\pi}{4}(\sqrt{2})$

$U_4 = \frac{\pi}{4}\left(2\left(\frac{1}{\sqrt{2}}\right) + 2(1)\right)$

$= \frac{\pi}{4}(\sqrt{2} + 2)$

2 a



$\Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}$

$x_i = x_0 + i\Delta x$

$\therefore$ $x_i = 1 + i\left(\dfrac{3}{n}\right)$

where $n = 1, 2, 3, ..., n$

b $m_i = f(x_{i-1}) = f\left(1 + \dfrac{3(i-1)}{n}\right)$

$\therefore$ $m_i = 2 + \dfrac{6(i-1)}{n}$

$\Rightarrow$ $L_n = \dfrac{3}{n}\displaystyle\sum_{i=1}^{n} m_i = \dfrac{3}{n}\left[\sum_{i=1}^{n} 2 + \dfrac{6}{n}\sum_{i=1}^{n}(i-1)\right]$

$= \dfrac{3}{n}\left[2n + \dfrac{6}{n}\,\dfrac{(n-1)n}{2}\right]$