Mathematical Exploration

"Exploring Sangaku Puzzles"

Introduction

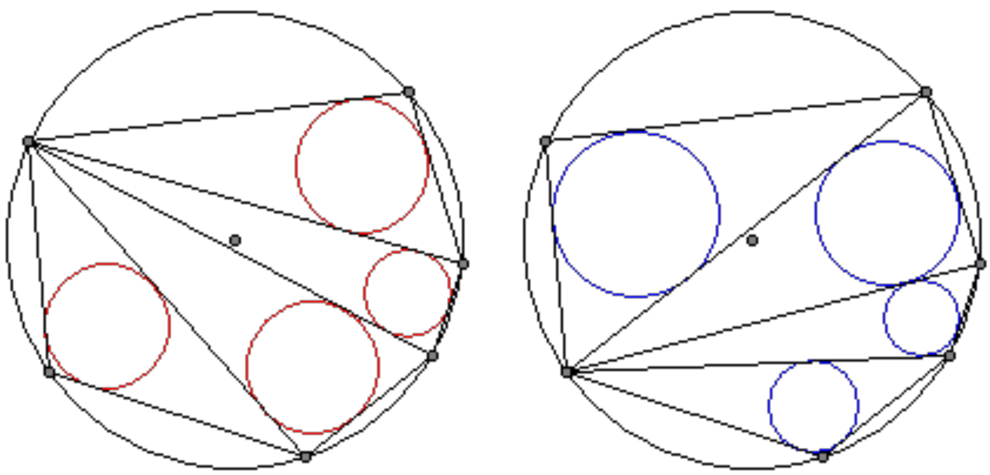
ABOUT SANGAKU:

Sangaku (算額) are Japanese geometrical puzzles in Euclidean geometry on wooden tablets.

Hundreds of years ago, people in Japan sacrifice cows and pigs for god. But this method is very expensive, so people carved pictures of animals onto wood as in replacement. One day, a Japanese samurai came up with the idea of offering god challenges. He started to draw some beautiful and creative patterns on the tablets. They are called Sangaku, which means mathematical tablets. Unfortunately, many of these tablets were lost during the period of modernization that followed the Edo period, but around nine hundred are known to remain.

The Dutch Japanologist Isaac Titsingh first introduced Sangaku to the West in the late 1970s. And now it already became very well known around the world.

Recently, Fukagawa and Tony Rothman from Princeton University wrote a book called "*Sacred Mathematics*". The book presents a detailed history about Sangaku, and also includes lots of Sangaku photos.



The diagram above shows a very typical Sangaku.

In a given circle, a polygon is drawn that every vertices is on the circle. Choose one vertices, and draw all the diagonals of the polygon that starts from that point, dividing the polygon into many triangles. Now, if we count the sum of radii of all the inscribed circles in those triangles, we will always get a same number. Which means, the sum of radii of the inscribed circles is not affected by the choice of vertices.

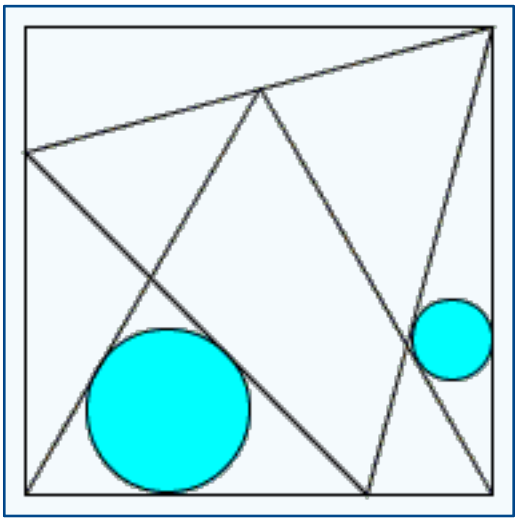
Most of the Sangaku puzzles give a theorem with a diagram to illustrate it, but usually the proofs are not given. Now, mathematicians proved many famous Sangaku, but there are still some unsolved ones.

AIM OF EXPLORATION:

From many of those interesting Sangaku puzzles, I chose one called "**Equilateral Triangles and Incircles in a Square**" to explore. I found that very interesting because it is constructed by 2 equilateral triangles and a square, we could find all sorts of triangles in the diagram, and also many pairs of related sides.

Here is the question:

Two equilateral triangles are inscribed into a square as shown in the diagram. Their side lines cut the square into a quadrilateral and a few triangles.



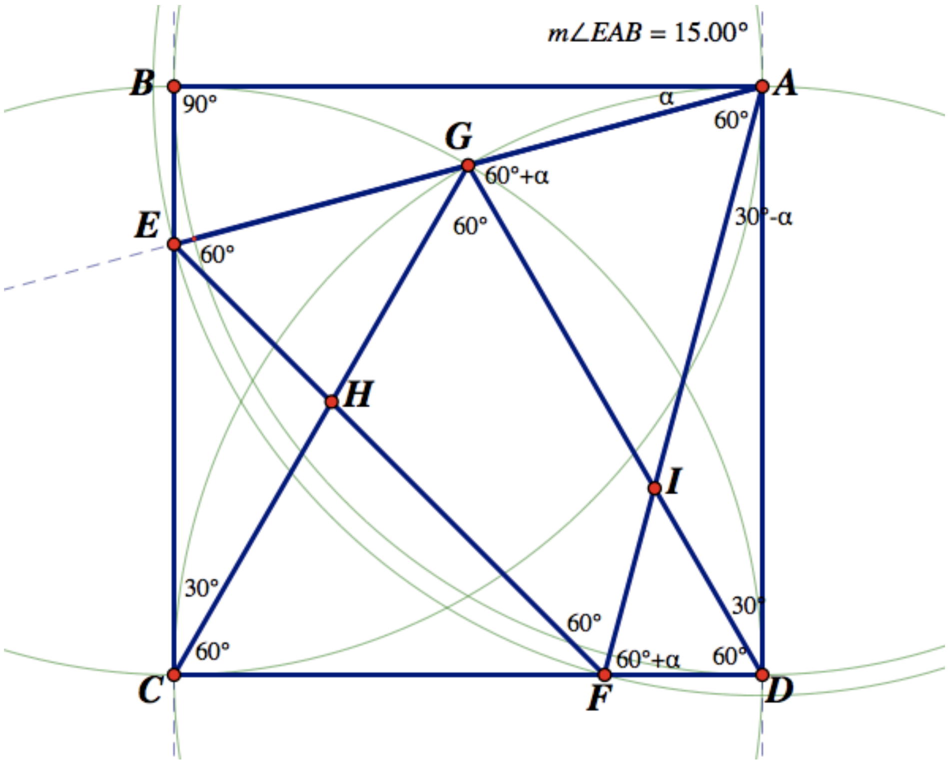
The aim of this exploration is to investigate the relationship between the radii of the incircles as shown in the diagram.

Investigation

ANGLES IN THE DIAGRAM:

This Sangaku has a very interesting shape. Two equilateral triangles are inscribed to a square, and divide the square into several triangles and a quadrilateral. If we look carefully, all sorts of triangles can be found in this diagram.

The diagram below is drawn using "Sketchpad" to help illustrating the question:



Square *ABCD* is drawn on the plain. Draw a circle with centre *C* and radius CD, and another circle with centre D and same radius. These two circles intersect at point G. ACG is an equilateral triangle.

Next, connect and extend AG, which meets BC at E. Rotate AE anticlockwise with centre A for 60°, which intersects with CD at point F. Then draw a circle with centre A and radius AF. As the program shows, the circle intersects with BC at point E. Thus, triangle AEF is equilateral.

Using the program, I measured the size of $\angle EAB$. As shown in the diagram, the answer is 15°.

$\angle EAB$ can also be calculated by analyzing the diagram, i.e.

Let $\alpha = \angle EAB$.

$\therefore \angle DAF = 90° - 60° - \alpha = 30° - \alpha$ ($\angle BAD = 90°$)

In ΔADG, $\angle ADG = 90° - 60° = 30°$

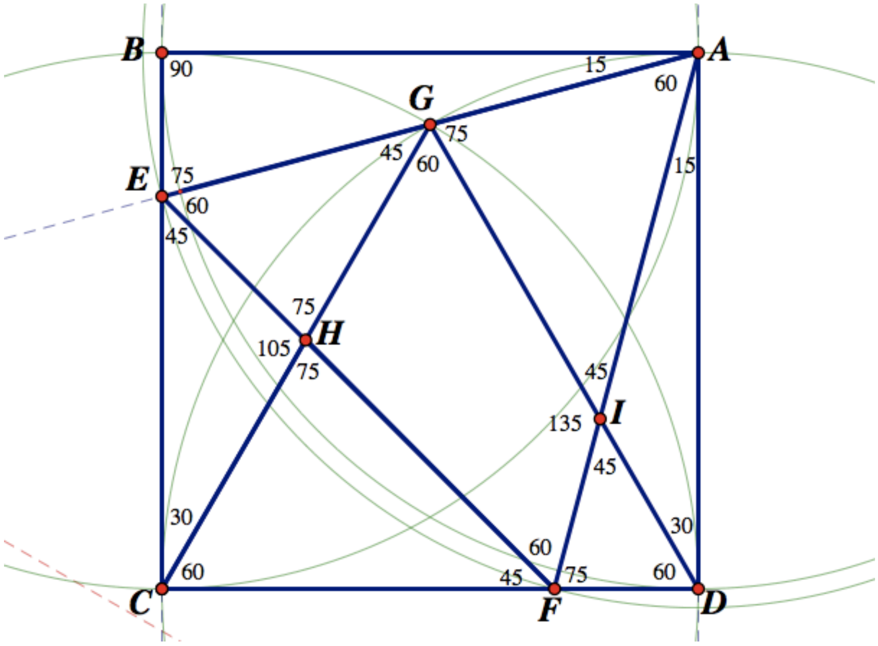
$\therefore \angle AGD = 180° - 30° - (60° + (30° - \alpha)) = 60° + \alpha$

And as ΔCDG is equilateral and $ABCD$ is a square, we know that $GD = CD = AD$

Which indicates that ΔADG is isolate.

$\therefore \angle DGA = \angle DAG \Rightarrow 60° + \alpha = 60° + (30° - \alpha)$
$\therefore 2\alpha = 30°$
$\therefore \alpha = 15°$

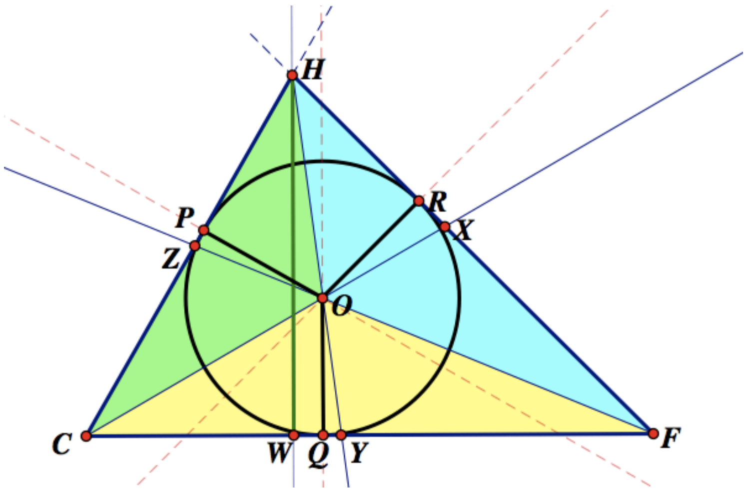
Now we can label every angle:



TRIANGLE AND ITS INSCRIBED CIRCLE

The question is about the relation between the inscribed circles of triangle CHF and triangle ADI. The centre of an inscribed circle is the intersection of 3 angle bisectors of the triangle, and the radius of the circle can be found by drawing a line from the centre that is perpendicular to one size of the triangle.

To investigate the relation between the triangle and its inscribed circle, I chose ΔCHF as an example.



In the diagram, line CX, HY and FZ are the angle bisectors of angle C, H and F. They intersect at point O. Draw $OP \perp HC$ at P, $OQ \perp CF$ at Q and $OR \perp FH$ at R. Then draw a circle with centre O and radius OP. Now we have inscribed circle of ΔCHF.

Let $OP = OQ = OR = r$

From the graph we can easily see that $S_{\Delta CHF} = S_{\Delta COH} + S_{\Delta HOF} = S_{\Delta FOC}$

In other words,

$$S_{\Delta CHF} = \frac{1}{2}(HC \times OP) + \frac{1}{2}(CF \times OQ) + \frac{1}{2}(FH \times OR)$$

$$= \frac{1}{2}r(HC + CF + FH)$$

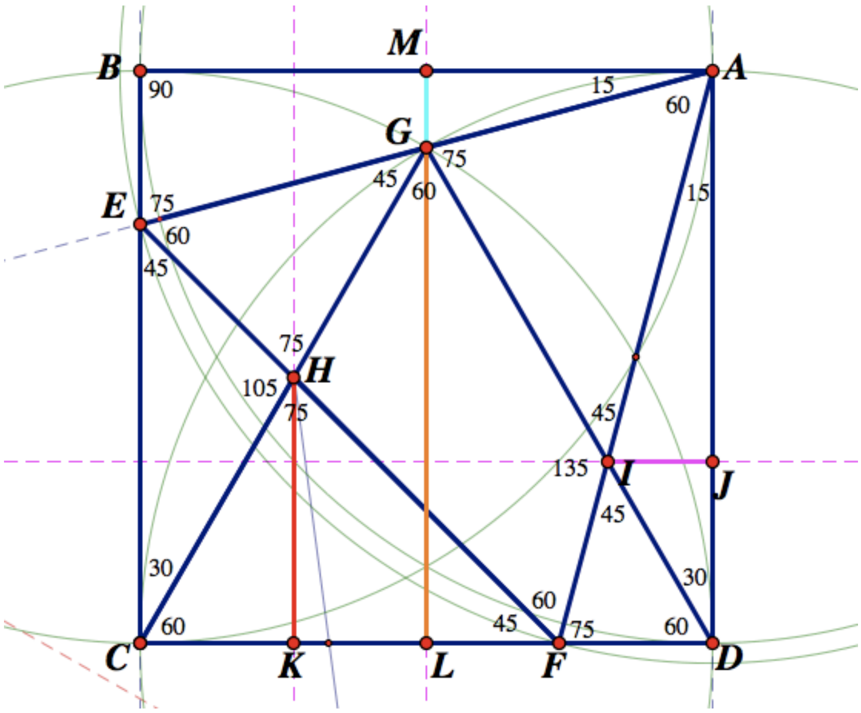
$$= \frac{1}{2}rp$$ (where p is the perimeter of the triangle)

$$\therefore r = \frac{2S}{p}$$

Thus, to find out the radius of the inscribed circle, we just need to find the area and the perimeter of the triangle.

AREA AND PERIMETER

To find out the area and perimeter of the two triangles, three lines are drawn in the diagram to help.



Draw $HK \perp CD$ at K, $GL \perp CD$ meets CD at L and meets AB at M, and $IJ \perp AD$ at J

Assume $AB = 2$, then each side of equilateral triangle CDG also have length 2.

$\therefore$ Height $GL = \frac{1}{2} CD \sin 60° = \sqrt{3}$

$\therefore MG = ML - GL = 2 - \sqrt{3}$

As $MG // BE$, $\Delta AMG \backsim \Delta ABE$

$\therefore BE = MG \times \frac{AB}{AM} = 2MG = 4 - 2\sqrt{3}$

Therefore using Pythagorean theorem,

$$AE = EF = FA = \sqrt{AB^2 + BE^2}$$
$$= \sqrt{2^2 + (4 - 2\sqrt{3})^2} = \sqrt{24 - 16\sqrt{3}} = \sqrt{8(\sqrt{3} - 1)^2}$$
$$= 2\sqrt{6} - 2\sqrt{2}$$

In ΔECF, $\angle CEF = \angle EFC \Rightarrow EC = FC$ and $\angle ECF = 90°$

$$\therefore CF = EF \sin 45° = \frac{2\sqrt{6} - 2\sqrt{2}}{\sqrt{2}} = 2\sqrt{3} - 2$$

And as $CF = CK + KF$ and $KF = KH$ ($\angle KHF = \angle KFH$),

$$2\sqrt{3} - 2 = CK + KH = CK + \cot 60° CK = (1 + \sqrt{3})CK$$

$$\therefore CK = \frac{2\sqrt{3} - 2}{1 + \sqrt{3}} = 4 - 2\sqrt{3}$$

$$CH = \frac{CK}{\sin 30°} = 2(4 - 2\sqrt{3}) = 8 - 4\sqrt{3}$$

$$KH = \frac{CK}{\cos 30°} = \sqrt{3}(4 - 2\sqrt{3}) = 4\sqrt{3} - 6$$

$$HF = \frac{KH}{\sin 45°} = \sqrt{2}(4\sqrt{3} - 6) = 4\sqrt{6} - 6\sqrt{2}$$

Thus we can calculate the area and perimeter of ΔCHF:

$$S_{\Delta CHF} = \frac{1}{2}(CF \times KH) = 18 - 10\sqrt{3}$$

$$p_{\Delta CHF} = CH + HF + FC = 6 - 6\sqrt{2} - 2\sqrt{3} + 4\sqrt{6}$$

As $AB = CD$ and $AE = AF$, we know that $\Delta ABE \cong \Delta ADF$

$$\therefore DF = BE = 4 - 2\sqrt{3}$$

And $IJ // DF$, so $\Delta AIJ \backsim \Delta AFD$

$$\therefore \frac{AJ}{IJ} = \frac{AD}{FD}$$

$$\Rightarrow \frac{AD - DJ}{IJ} = \frac{AD}{FD}$$

$$\Rightarrow \frac{2 - \cot 30° IJ}{IJ} = \frac{2}{4 - 2\sqrt{3}} \Rightarrow \frac{2 - \sqrt{3}IJ}{IJ} = \frac{2}{4 - 2\sqrt{3}}$$

$$\therefore \frac{2}{IJ} = \frac{2}{4 - 2\sqrt{3}} + \sqrt{3}$$

$$IJ = \frac{\sqrt{3} - 1}{2}$$

$$AI = IJ \times \frac{AF}{FD} = \frac{\sqrt{3}-1}{2} \times \frac{2\sqrt{6}-2\sqrt{2}}{4-2\sqrt{3}} = \sqrt{2}$$

$$ID = \frac{IJ}{\sin 30°} = 2(\frac{\sqrt{3}-1}{2}) = \sqrt{3}-1$$

Thus we can calculate the area and perimeter of ΔAID:

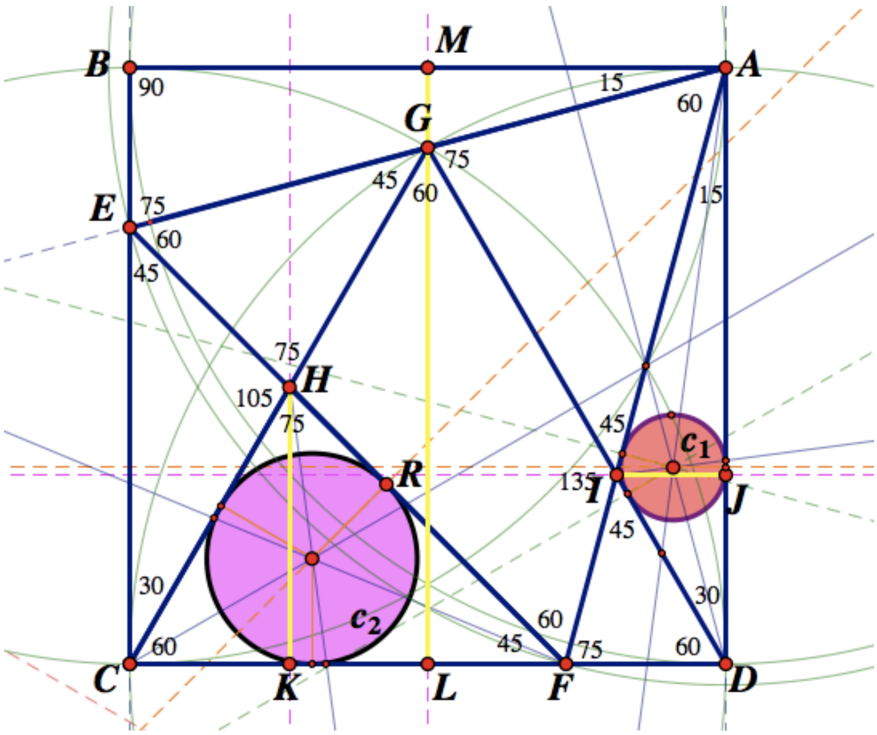
$$S_{\Delta AID} = \frac{1}{2}(AD \times IJ) = \frac{\sqrt{3}-1}{2}$$

$$p_{\Delta AID} = AI + ID + DA = 1 + \sqrt{2} + \sqrt{3}$$

RADII OF INSCRIBED CIRCLES

As mentioned before in "Triangle and its Inscribed circle", the radius of the inscribed circle of a triangle can be calculated by:

$$r = \frac{2S}{p}$$



We can see from the diagram above, c_1 is inscribed in ΔAID and c_2 is inscribed in ΔCHF.

$$\therefore r_1 = \frac{2S_{\Delta AID}}{p_{\Delta AID}} = \frac{\sqrt{3}-1}{1+\sqrt{2}+\sqrt{3}}$$

$$\therefore r_2 = \frac{2S_{\Delta CHF}}{p_{\Delta CHF}} = \frac{36-20\sqrt{3}}{6-6\sqrt{2}-2\sqrt{3}+4\sqrt{6}}$$

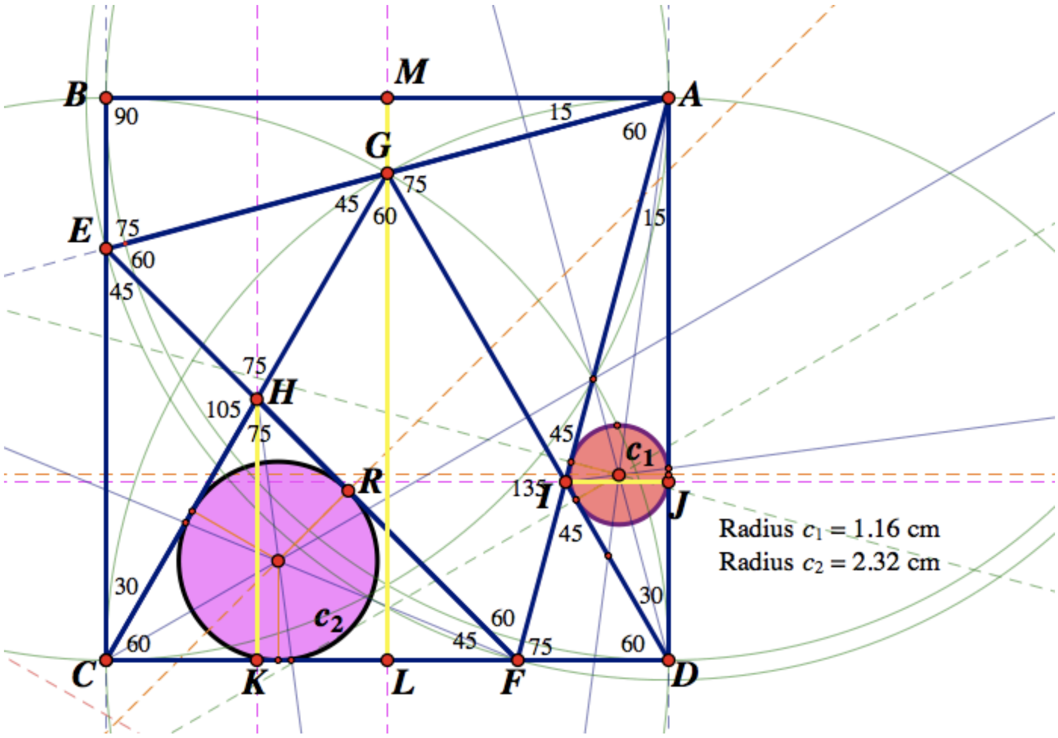
$$\therefore r_1 : r_2 = \frac{\sqrt{3}-1}{1+\sqrt{2}+\sqrt{3}} \div \frac{36-20\sqrt{3}}{6-6\sqrt{2}-2\sqrt{3}+4\sqrt{6}}$$

$$= \frac{6\sqrt{3}-6\sqrt{6}-6+12\sqrt{2}-6+6\sqrt{2}+2\sqrt{3}-4\sqrt{6}}{36\sqrt{3}+36\sqrt{2}+36-60-20\sqrt{6}-20\sqrt{3}}$$

$$= \frac{-12+18\sqrt{2}+8\sqrt{3}-10\sqrt{6}}{-24+36\sqrt{2}+16\sqrt{3}-20\sqrt{6}}$$

$$= \frac{1}{2}$$

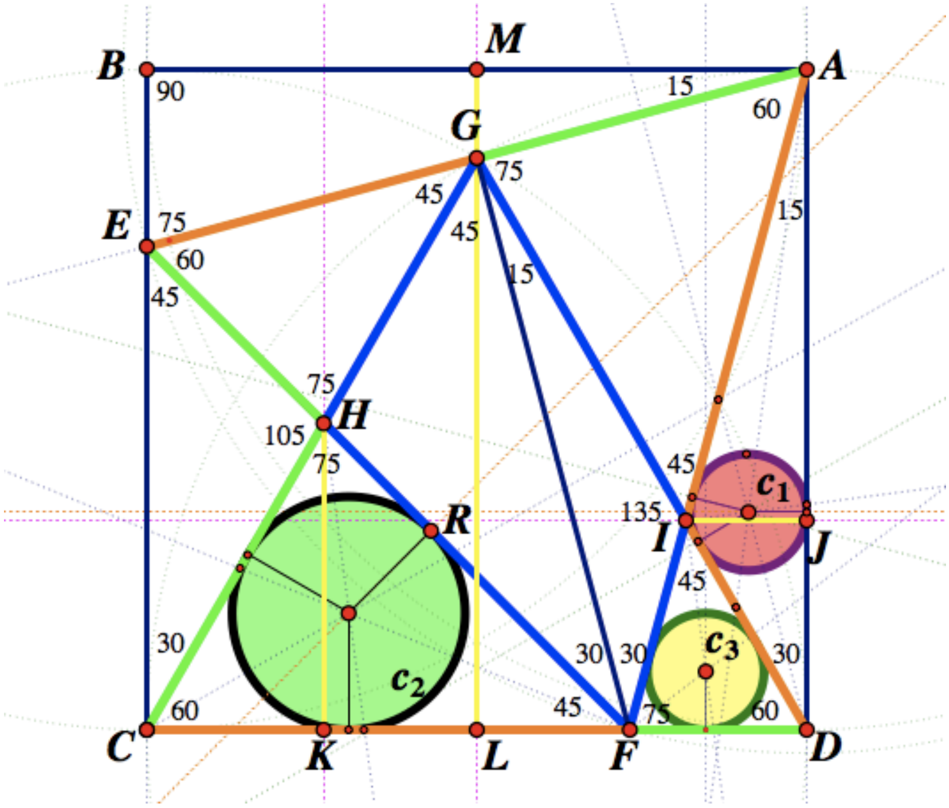
As we found out by above calculations, the ratio of two radii is $1:2$. This can be confirmed with the help of technology. In Sketchpad, I measured the length of two radii:



Hence we know that $r_1 : r_2 = 1.16 : 2.32 = 1 : 2$, and so we came to the same conclusion.

MORE FINDINGS

After finding the relation between the two radii, I looked at the diagram again, and found some very interesting relations between the triangles.



From the angles that are labeled in the diagram, we can find many similar triangles, i.e.

$\Delta CHF \backsim \Delta DFI \backsim \Delta AGI \backsim \Delta EGH$

Angles in both triangles are $60°$, $75°$ and $45°$.

And to see the relation clearer, I colored each pair of corresponding sides in same color, as shown above.

Also, if we connect GF, we can also find that $\Delta GIF \backsim \Delta AID$ and $\Delta EHC \backsim \Delta GHF$

What is more interesting is that these triangles together form two equilateral triangles, ΔAEF and ΔCDG. So we can find 2 pairs of equal sides:

$$\begin{cases} CH + HG = GI + ID = FD + CF = 2 \\ EG + GA = AI + IF = FH + HE = 2\sqrt{6} - 2\sqrt{2} \end{cases}$$

Moreover, I looked through the previous calculations about the length of each segment, and what I found was that $CF = 2ID$ ($CF = 2\sqrt{3} - 2$ and $ID = \sqrt{3} - 1$), so this indicates that $r_3 = \frac{1}{2} r_2 = r_1$, since c_3 is inscribed in ΔDFI and $\Delta CHF \backsim \Delta DFI$, thus circle c_1 and c_3 has same radius.

Reflection

By doing this exploration on Sangaku Puzzle, I've learned many things, i.e.

- The technique of converting $\tan 15°$ into an irrational number (the length of segment MG). It will be very useful when solving a geometry problem without a calculator.

- Knowledge about inscribed circle in a triangle, such as the place of the centre and the formula $rp = 2S$.

- The way to use the geometric software "Sketchpad"– using the software I am able to draw very accurate graphs and take precise measurements. It made the question clearer and easier to solve, and also saved my time.

- Calculating skills, especially division and multiplication involving irrational numbers.

For extension, we could investigate the relations between other triangles in the diagram (they are mentioned in "More Findings"). For example, we can draw inscribed circles in every triangle and study the ratio of their radii.

Also, questions are raised for further investigate:

Can we convert other trigonometric functions into irrational numbers using similar method?

Can we prove the relationship between r_1, r_2 and r_3 without using the formula "$rp = 2S$"?

Will this relation still remains the same if we let point A slides on ray DA (where G, F still on AE, CD and AEF forms an equilateral triangle)? Why?

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