IB Math HL2: Converging and Diverging Sequences **Name:** ____________ **Period:** ______

A sequence is defined a function whose domain is the set of positive integers and denoted $\{a_n\}$. $n \in \mathbb{Z}^+$

A sequence converges to the limit L when the limit exists. Otherwise the sequence diverges. Generally a sequence will diverge if the sequence increases or decreases without bound or if the terms do not "target in" on one specific value.

1. Does the following sequence converge? If it does, find the limit of convergence. $\left\{\dfrac{n^2}{2^n - 1}\right\}$.

$$\lim_{n\to\infty} \frac{n^2}{2^n - 1} \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{2n}{(\ln 2)\cdot 2^n} \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{2}{(\ln 2)^2 2^n} = \frac{2}{\infty} = 0$$

$$\boxed{\therefore \left\{\frac{n^2}{2^n-1}\right\} \text{ converges to } 0}$$

2. The following is some terminology associated with sequence $\{a_n\}$.

Name	Condition
Strictly Increasing	$a_1 < a_2 < \ldots\ldots < a_{k-1} < a_k \ldots.$
Increasing	$a_1 \le a_2 \le \ldots\ldots \le a_{k-1} \le a_k \ldots..$
Strictly Decreasing	$a_1 > a_2 > \ldots\ldots > a_{k-1} > a_k \ldots..$
Decreasing	$a_1 \ge a_2 \ge \ldots\ldots \ge a_{k-1} \ge a_k \ldots..$
Bounded above by M	$a_n \le M$ for n=1,2,3,……… ∞
Bounded below by m	$a_n \ge m$ for n=1,2,3, ………. ∞
Bounded	If it is bounded both above and below

$\dfrac{da_n}{dn} > 0$

$\dfrac{da_n}{dn} < 0$

$\lim\limits_{n\to\infty} a_n$

A sequence is monotonic if it is increasing or decreasing or strictly monotonic if it is strictly increasing or strictly decreasing.

The bounded, monotonic, convergence theorem (BMCT)

A monotonic sequence $\{a_n\}$ converges if it is bounded and diverges otherwise.

3. Show that the sequence converges by showing it is either increasing with an upper bound or decreasing with a lower bound.

a. $\left\{\dfrac{\ln n}{\sqrt{n}}\right\}$ b. $\left\{\dfrac{3n-2}{n}\right\}$ c. $\left\{\dfrac{3n-1}{2^n}\right\}$

✓ 1) Show Increasing/Decreasing

$$\frac{d}{dn} a_n \Rightarrow \oplus \Rightarrow \text{Increasing}$$
$$\ominus \Rightarrow \text{Decreasing}.$$

✓ 2) Find the bound.

$$\lim_{n\to\infty} a_n = L \begin{cases} \text{Increasing with upper Bound} \\ \text{Decreasing with Lower Bound.} \end{cases}$$

a) $a_n = \left\{ \frac{\ln n}{\sqrt{n}} \right\}$

$$\left(\begin{array}{ll} n=e & n=e^2 \\ \frac{\ln e}{\sqrt{e}} = \frac{1}{\sqrt{e}} & \frac{2}{\sqrt{e^2}} = \frac{2}{e} \end{array} \right)$$

1) $$\frac{d}{dn}\left(\frac{\ln n}{\sqrt{n}}\right) = \frac{\left(\frac{1}{n}\right)\cdot\sqrt{n} - \frac{1}{2\sqrt{n}}\cdot \ln n}{(\sqrt{n})^2}$$

$$= \frac{\left(\frac{1}{\sqrt{n}} - \frac{\ln n}{2\sqrt{n}}\right)\cdot 2\sqrt{n}}{(n)\ 2\sqrt{n}}$$

$2 - \ln n = 0$

$n = e^2$

$$= \frac{2 - \ln n}{2\sqrt{n}\cdot n} < 0$$

for $n > e^2$

2) $$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{2\sqrt{n}}{n} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{1} = \frac{1}{\infty} = \boxed{0}$$

$\therefore \left\{ \frac{\ln n}{\sqrt{n}} \right\}$ is decreasing with Lower Bound of 0.

b) $\left\{ \dfrac{3n-2}{n} \right\}$.

1) Increasing / Decreasing.

$$f(n) = \frac{3n-2}{n} \Rightarrow \frac{df}{dn} = \frac{3n-(3n-2)}{n^2} = \boxed{\frac{2}{n^2} > 0}$$

for $n \in \mathbb{Z}^+$.

2) Limit:

$$\lim_{n\to\infty} \frac{3n-2}{n} = \lim_{n\to\infty} \frac{3}{1} = \boxed{3}$$

$\therefore \left\{ \dfrac{3n-2}{n} \right\}$ Increases with U.B of 3.

c) $\left\{ \dfrac{3n-1}{2^n} \right\}$

1) Increasing/ Decreasing.

$\therefore \left\{ \dfrac{3n-1}{2^n} \right\}$ decreases with L.B of 0

$$f(n) = \frac{3n-1}{2^n} \Rightarrow \frac{df}{dn} = \frac{3(\cancel{2^n}) - (3n-1)\cdot \ln 2 \cdot \cancel{2^n}}{(2^n)^{\cancel{2}}}$$

$A - B\cdot n$

$$= \frac{3-(3n-1)\cdot \ln 2}{2^n} = \frac{3+\ln 2 - (\ln 2)(3n)}{2^n} < 0$$

for $\left(n > \dfrac{3+\ln 2}{3(\ln 2)} \right)$

2) $$\lim_{n\to\infty} \frac{3n-1}{2^n} \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{3}{(\ln 2)(2^n)}$$

$$= \frac{3}{\infty} = 0$$