7B Questions Answers. ①

#1. Ratio test

a) $\lim\limits_{n\to\infty}\left(\frac{n x^{n+1}}{(n+1)^2 2^{n+1}}\right)\left(\frac{n^2 2^n}{(n-1)x^n}\right) = \lim\limits_{n\to\infty}\frac{n^3}{(n+1)^2(n-1)}\cdot\frac{x}{2}$ $\quad \div n^3$

$= \lim\limits_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}}\right)\cdot\frac{x}{2} = \frac{x}{2}$ $\quad \boxed{\frac{R}{2}=1}$

$-2 < x < 2$

$\boxed{R=2}$

b) When $x = 2$

$\sum \frac{(n-1)}{n^2}$

Use limit comparison $b_n = \frac{1}{n}$.

$\Rightarrow \lim\limits_{n\to\infty}\frac{\frac{n-1}{n^2}}{\frac{1}{n}} = \lim\limits_{n\to\infty}\frac{n^2-n}{n^2}\;\frac{\div n^2}{\div n^2} = \lim\limits_{n\to\infty}\frac{1-\frac{1}{n}}{1} = 1$

$\sum \frac{1}{n}$ diverge $\Leftrightarrow \int_1^\infty \frac{1}{x}\,dx = \lim\limits_{a\to\infty}\left[\ln x\right]_1^a$

$= \lim\limits_{a\to\infty}(\ln a - 0) = \infty$

$\therefore \sum \frac{n-1}{n^2}$ diverges.

When $x = -2$

$\sum \frac{(n-1)}{n^2}(-1)^n$

we know $\sum \frac{(n-1)}{n^2}$ diverges $\Big)$ $\Rightarrow$ conditional convergent series.

$\therefore$ Interval of convergence $\boxed{[-2, 2)}$

Alt Series test

$\lim\limits_{n\to\infty}\frac{n-1}{n^2} = 0 \qquad \frac{n-1}{n^2}$ is decreasing: $\left(\frac{n-1}{n^2}\right)' = \frac{n^2-2n(n-1)}{n^4}$ < 0

\# 2

Total [19 marks]

X (a) (i) $f(x) = (1+ax)(1+bx)^{-1}$

$= (1+ax)(1-bx+...(-1)^n b^n x^n + ...$ *M1A1*

it follows that

$c_n = (-1)^n b^n + (-1)^{n-1} ab^{n-1}$ *M1A1*

$= (-b)^{n-1}(a-b)$ *AG*

(ii) $R = \dfrac{1}{|b|}$ *A1*

[5 marks]

(b) to agree up to quadratic terms requires

$1 = -b + a,\ \dfrac{1}{2} = b^2 - ab$ *M1A1A1*

from which $a = -b = \dfrac{1}{2}$ *A1*

[4 marks]

(c) $e^x \approx \dfrac{1+0.5x}{1-0.5x}$ *A1*

putting $x = \dfrac{1}{3}$ *M1*

$e^{\frac{1}{3}} \approx \dfrac{\left(1+\frac{1}{6}\right)}{\left(1-\frac{1}{6}\right)} = \dfrac{7}{5}$ *A1*

[3 marks]

Total [12 marks]

③

#3. $\frac{1}{1\times 2} + \frac{1}{4\times 5}x + \frac{1}{7\times 8}x^2 + \frac{1}{10\times 11}x^3 \cdots$

(a) $b(n) = \overset{n=0}{1}, \overset{n=1}{4}, \overset{n=2}{7}, 10 \cdots$

$= 3n+1$

$c(n) = 2, 5, 8, 11 \cdots$

$= 3n+2$

(b) $\sum_{n=0}^{\infty} \frac{1}{(3n+2)(3n+1)} x^n$

Ratio test. x.

$\lim_{n\to\infty} \left[\frac{x^{n+1}}{(3(n+1)+2)(3(n+1)+1)}\right]\left[\frac{(3n+2)(3n+1)}{x^n}\right] = x$

$|x| < 1 \Rightarrow -1 < x < 1$ $\boxed{R = 1}$

(c) when $x = 1$

$\sum \frac{1}{(3n+2)(3n+1)} < \sum \frac{1}{n^2}$

$\Rightarrow \sum \frac{1}{n^2}$ converges ~~by test test~~

by p series $p = 2$

#4. $f(x)=\sqrt{x} = x^{\frac{1}{2}}$

(a) $f(x)=1$

$f'(x)=\frac{1}{2}(x)^{-\frac{1}{2}} \Rightarrow f'(1)=\frac{1}{2}$

$f''(x)=-\frac{1}{4}(x)^{-\frac{3}{2}} \Rightarrow f''(1)=\frac{-1}{4}$

$f'''(x)=\frac{3}{8}(x)^{-\frac{5}{2}} \Rightarrow f'''(1)=\frac{3}{8}$.

$$\Rightarrow \sqrt{x} = 1+\frac{1}{2}(x-1)+\left(\frac{-1}{4}\right)(x-1)^2+\frac{3}{8}(x-1)^3 \ldots$$

(b) $$\lim_{x\to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)} = \lim_{x\to 1}\frac{\cancel{x-1}}{(\cancel{x-1})(\sqrt{x}+1)}$$

$$= \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \boxed{\frac{1}{2}}$$

#5. (a) $$\int_2^\infty \frac{dx}{x(\ln x)^k}$$

$\ln x = u$ $\quad x=2 \quad u=\ln 2$

$du=\frac{1}{x}dx \quad x\to\infty \quad u\to\infty$

$$= \int_{\ln 2}^\infty \frac{du}{u^k}$$

$$= \left[\frac{1}{1-k}(u)^{1-k}\right]_{\ln 2}^\infty \Rightarrow$$

$k=1. \Rightarrow$ diverges

$k>1 \Rightarrow$ converse.

$k<1 \Rightarrow$ diverges.

$k>1 \Rightarrow$ converges

$k\le 1 \Rightarrow$ diverges.

#5. (b) $\displaystyle\sum_{r=2}^{\infty} \left| \frac{(-1)^r}{r \ln r} \right|$

$\Rightarrow \displaystyle\sum_{r=2}^{\infty} \frac{1}{r \ln r} \Rightarrow$ diverges Shown at (a)

$k=1$.

$\Rightarrow \displaystyle\sum_{r=2}^{\infty} \frac{(-1)^r}{r \ln r}. \Rightarrow$ Alt. Series

$$\lim_{r \to \infty} \frac{1}{r \ln r} = 0$$

$$\frac{1}{r \ln r} > \frac{1}{(r+1)\ln(r+1)} \Rightarrow \text{decreasing.}$$

$\therefore \displaystyle\sum_{r=2}^{\infty} \frac{(-1)^r}{r \ln r}$ Conditionally converges.

#6. (a)

$dx = 0.1$ $\qquad dy = \left(\frac{y^2}{1+x}\right) dx$

x	y	dy
0	1	0.1
0.1	1.1	0.11
0.2	1.21	
0.3	1.332	
0.4	1.4685	
0.5	$\boxed{1.62}$	

$\boxed{(0.5,\ 1.62)}$

#6. (b) $$\frac{d^2y}{(dx)^2} = \frac{\left(2y \cdot \frac{dy}{dx}\right)(1+x) - y^2(1)}{(1+x)^2}$$

$$= \frac{(2y)\left(\frac{y^2}{1+\cancel{x}}\right)(1\cancel{+x}) - y^2}{(1+x)^2} = \frac{2y^3 - y^2}{(1+x)^2} = \frac{d^2y}{(dx)^2}$$

6. (ii) $x=0,\ y=1$ $y' = \frac{1}{1+0} = 1$ $y'' = \frac{2-1}{1} = 1$ (5)

$$\boxed{M(x) = 1 + x + \frac{x^2}{2!}}$$

(b) (i) $\frac{dy}{dx} = \frac{y^2}{1+x} \Rightarrow \frac{dy}{y^2} = \frac{dx}{1+x}$

$$\frac{-1}{y} = \ln|1+x| + C.$$

$x=0,\ y=1$

$C = -1$

$$\frac{-1}{y} = \ln|1+x| - 1$$

$$\frac{1}{y} = 1 - \ln|1+x|.$$

$$\boxed{y = \frac{1}{1 - \ln|1+x|}}$$

(ii) V. A $\Rightarrow$ $1 - \ln|1+x| \Rightarrow \ln|1+x| = 1$

$1+x = e$ $\boxed{x = e-1}$