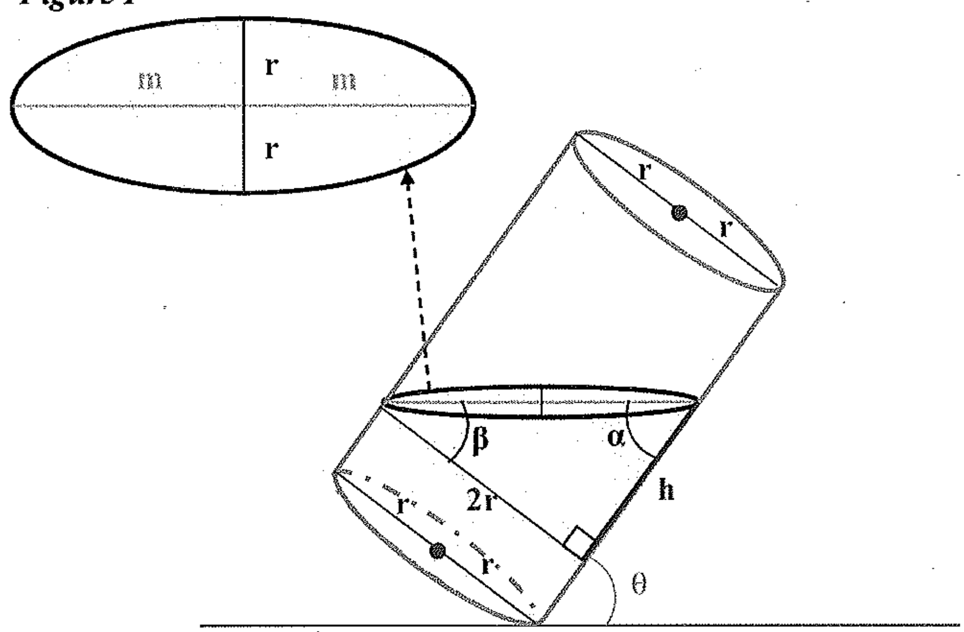
17 May 2012

Exploration: Surface area of water in a tilted cylinder

In this exploration, I will explain how tilting a cylinder affects the surface area of water inside the cylinder. Given the stimulus word "water", I started looking into aspects of water that involved modeling and real-world application, because I prefer to work with real ideas rather than abstract concepts. I first became interested in this topic when I noticed that tipping a glass of water caused the water to "spread out" parallel to the ground, and the surface area increased. At first, I wanted to investigate changing several variables, such as how the angle of the cup's walls affected the surface area of the water as the cup tipped, but it quickly became apparent that that topic would be much too big for a single exploration. I simplified the concept down to an upright cylinder, whose sides were perpendicular to its circular base.

To determine the formula that related the angle between the side of the cylinder and the ground to the area of the ellipse created by the surface area of the water inside the cylinder, I first used well-known trigonometric angles, and looked for a pattern. The diagram below (*Figure 1*) demonstrates where each of the variables is used:

Figure 1



r = radius of the cylinder (independent variable)
h = distance along the side of the cylinder from each of the extremes of the ellipse formed by the surface of the water in the cylinder (dependent variable)
θ = changing angle (independent variable)
α = θ (using parallel lines)
β = 90° - α (using properties of a triangle)

Note: *Figure 1* is color-coded. For example, each segment that is green is the same length.

The relationship between θ, m, and r is as follows:

$$m = r/\sin\theta$$

Table 1

θ (°)	sin θ	m	r/sin θ
30	$\frac{1}{2}$	$2r$	$2r$
45	$\frac{1}{\sqrt{2}}$	$r\sqrt{2}$	$r\sqrt{2}$
60	$\frac{\sqrt{3}}{2}$	$2r\sqrt{3}$	$2r\sqrt{3}$
90	1	r	r

Looking at the model, this makes sense. If I take just the right triangle formed with the sides *h*, *2r*, and *2m*:

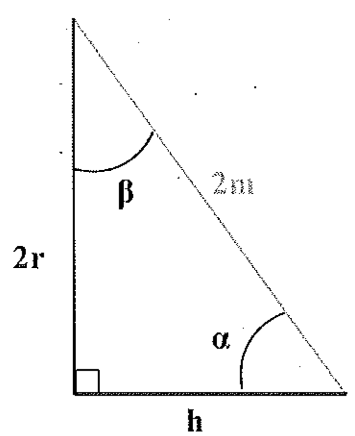


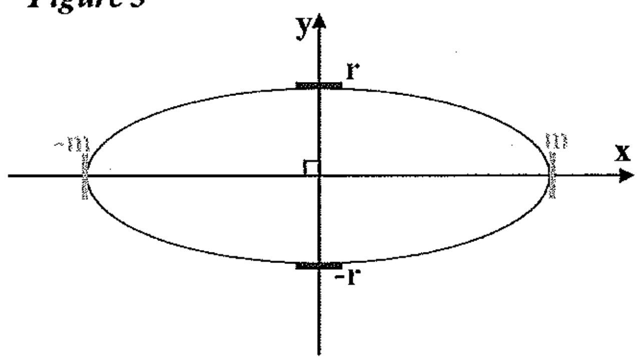
Figure 2

As shown by *Figure 2*, above, $\sin\alpha = 2r/2m = r/m$. Therefore, $m = r/\sin\alpha$. Since $\alpha = \theta$ (using parallel lines, as seen in *Figure 1*), $m = r/\sin\theta$.

The equation for an ellipse is:

$$\frac{x^2}{m^2}+\frac{y^2}{r^2}=1$$

Figure 3



m is the longer of the two axes, because *m* is the axis being stretched, while *r* remains constant. The minimum value for *m* is equal to *r*. *r* is the radius of the cylinder.

Making *y* the subject of the formula:

$$\frac{y^2}{r^2}=1-\frac{x^2}{m^2}$$

$$y=\pm\sqrt{\frac{r^2(m^2-x^2)}{m^2}}$$

$$y=\pm\frac{r}{m}\sqrt{m^2-x^2}$$

To find the area of the ellipse, integrate the equation with respect to x (see *Figure 3*):

$$A=\int y\,dx=\int\pm\frac{r}{m}\sqrt{m^2-x^2}\,dx$$

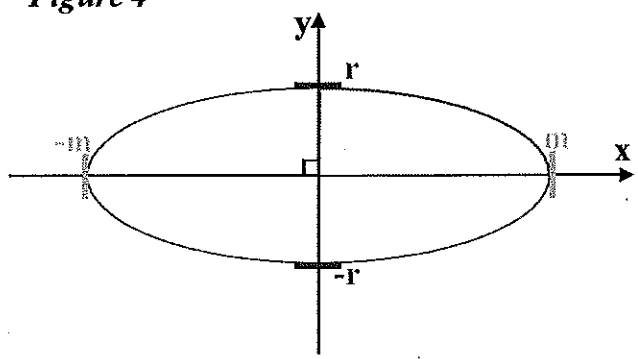
Notice that the area of an ellipse centered at the origin is equal to 4 times the area of the integral of the equation in the first quadrant. Therefore, since $x = m$ is the x-intercept in the first quadrant:

$$A=4\int_0^m\frac{r}{m}\sqrt{m^2-x^2}\,dx$$

$$A=4\frac{r}{m}\int_0^m\sqrt{m^2-x^2}\,dx$$

(As seen in *Figure 4*, below)

Figure 4



Let $x = m \sin \varphi$. Therefore, $dx = m \cos \varphi \, d\varphi$, and the limits of integration change:

$0 = m \sin \varphi$	$m = m \sin \varphi$
$\varphi = 0$	$\varphi = \pi/2$

$$A = 4\frac{r}{m}\int_0^{\frac{\pi}{2}} \sqrt{m^2 - (m \sin \varphi)^2}\,(m \cos \varphi)\, d\varphi$$

$$A = 4\frac{r}{m}\int_0^{\frac{\pi}{2}} m\sqrt{1 - (\sin \varphi)^2}\,(m \cos \varphi)\, d\varphi$$

$$A = 4\frac{r}{m}\int_0^{\frac{\pi}{2}} m^2 \cos \varphi \sqrt{(\cos \varphi)^2}\, d\varphi$$

$$A = 4rm\int_0^{\frac{\pi}{2}} (\cos \varphi)^2\, d\varphi$$

Since $(\cos \theta)^2 = \frac{1+\cos 2\varphi}{2}$, this can be substituted:

$$A = 4rm\int_0^{\frac{\pi}{2}} \frac{1+\ \cos 2\varphi}{2}\, d\varphi$$

$$A = 2rm\int_0^{\frac{\pi}{2}} 1 +\ \cos 2\varphi\, d\varphi$$

$$A = 2rm\left[\varphi + \frac{1}{2}\sin 2\varphi\right]_0^{\frac{\pi}{2}}$$

$$A = 2rm\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin \pi\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$$

$$A = 2rm\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right]$$

$$A = 2rm\left(\frac{\pi}{2}\right)$$

$$A = \pi rm$$

Assuming that is true, the surface area of the water relative to angle θ would be:

$$A = \pi m r$$
$$A = \pi\left(\frac{r}{\sin\theta}\right)(r)$$
$$A = \frac{\pi r^2}{\sin\theta}$$

When θ = 0°, there is a divide-by-zero error, because sin (0°) = 0. This makes sense in the context of the application: When the cylinder is flat on its side, the water will flow out of the cylinder.

It is important to note that this model assumes that the thickness of the edges of the cylinder itself is negligible. Of course, the angle θ would have to be calculated between a tangent to the bottommost corner of the cylinder which is parallel to the ground and the interior line of the side of the cylinder, rather than simply between the ground and the outside edge of the cylinder.

Another significant limitation is that this particular formula only works if the cylinder is a right cylinder with a circular base. If the sides of the cylinder are askew, or tilted in any way, or if the base is elliptical, this formula no longer produces accurate results.

It is interesting to note that if the cylinder is tilted past 90° and the angle θ becomes obtuse, the formula still works. Because the sine function gives positive results for all values of θ greater than 0° and less than 180°, the surface area produced by the formula is still accurate. This is observable because if the cylinder is tipped left instead of right (as seen above in *Figure 1*), the surface area is the same for the acute angle and the same angle reflected over 90°. It doesn't matter which way the cylinder is tipped, the surface area is the same for the angle.

Below (in *Table 2*) are some example values using the formula:

Table 2

θ (°)	**radius of the cylinder (cm)**	**sin θ**	**Surface area of the water in the cylinder (cm^2)**
30	10	$\frac{1}{2}$	200π
30	20	$\frac{1}{2}$	800π
30	30	$\frac{1}{2}$	1800π
30	40	$\frac{1}{2}$	3200π

30	50	$\frac{1}{2}$	5000π
45	10	$\frac{1}{\sqrt{2}}$	100π x √2 = 141.4π
45	20	$\frac{1}{\sqrt{2}}$	400π x √2 = 565.7π
45	30	$\frac{1}{\sqrt{2}}$	900π x √2 = 1272.8π
45	40	$\frac{1}{\sqrt{2}}$	1600π x √2 = 2262.7π
45	50	$\frac{1}{\sqrt{2}}$	2500π x √2 = 3535.5π
60	10	$\frac{\sqrt{3}}{2}$	200π / √3 = 115.5π
60	20	$\frac{\sqrt{3}}{2}$	800π / √3 = 461.9π
60	30	$\frac{\sqrt{3}}{2}$	1800π / √3 = 1039.2π
60	40	$\frac{\sqrt{3}}{2}$	3200π / √3 = 1847.5π
60	50	$\frac{\sqrt{3}}{2}$	5000π / √3 = 2886.8π
90	10	1	100π
90	20	1	400π
90	30	1	900π
90	40	1	1600π
90	50	1	2500π
120	10	$\frac{1}{2}$	200π
120	20	$\frac{1}{2}$	800π
120	30	$\frac{1}{2}$	1800π
120	40	$\frac{1}{2}$	3200π
120	50	$\frac{1}{2}$	5000π
135	10	$\frac{1}{\sqrt{2}}$	100π x √2 = 141.4π
135	20	$\frac{1}{\sqrt{2}}$	400π x √2 = 565.7π
135	30	$\frac{1}{\sqrt{2}}$	900π x √2 = 1272.8π
135	40	$\frac{1}{\sqrt{2}}$	1600π x √2 = 2262.7π
135	50	$\frac{1}{\sqrt{2}}$	2500π x √2 = 3535.5π

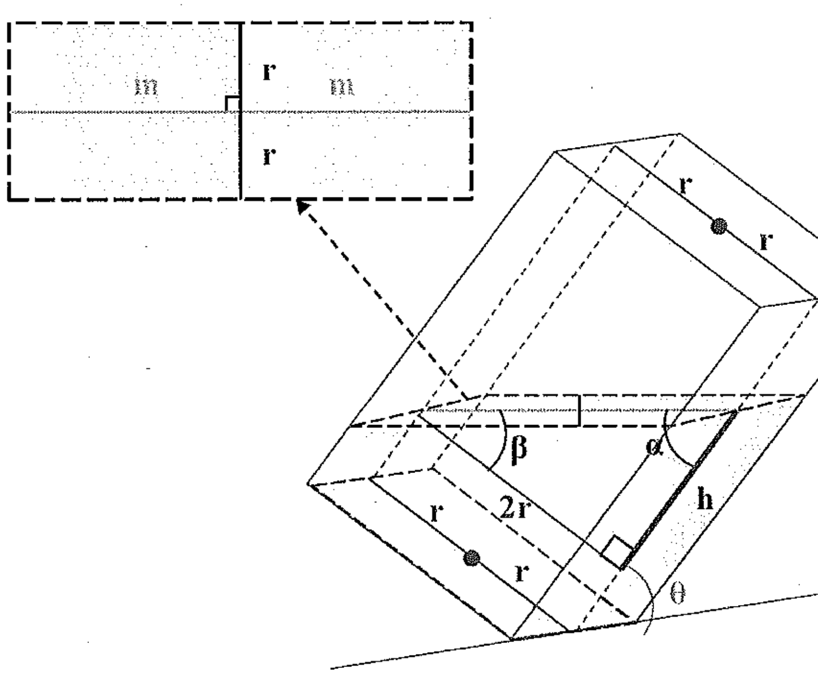
150	10	$\frac{\sqrt{3}}{2}$	$200\pi / \sqrt{3} = 115.5\pi$
150	20	$\frac{\sqrt{3}}{2}$	$800\pi / \sqrt{3} = 461.9\pi$
150	30	$\frac{\sqrt{3}}{2}$	$1800\pi / \sqrt{3} = 1039.2\pi$
150	40	$\frac{\sqrt{3}}{2}$	$3200\pi / \sqrt{3} = 1847.5\pi$
150	50	$\frac{\sqrt{3}}{2}$	$5000\pi / \sqrt{3} = 2886.8\pi$

Note that the values for θ are the same as the values for 180° - θ, because $\sin\theta = \sin(180° - \theta)$.

The values in Table 2 are only listed as accurate to one decimal place because having more than that seemed excessive, and hinders the readability of the table.

This same concept can be applied to containers that have other bases than just a circle. For example, the surface area of the water in a square-based container could be calculated using similar values. Instead of $2r$ representing the minor axis of the ellipse and $2m$ representing the major axis of the ellipse, they can be reassigned to represent the minor and major side lengths of a rectangle, respectively, as shown in *Figure 5*, below.

Figure 5



Using a similar formula to that of the cylinder, we can see that the relationship between θ, m, and r is as follows:

$$m = r/\sin\theta$$

Hence, we can calculate the surface area of the water in the container. Since it is known that the area of a rectangle is $A = bh$, and we can see from *Figure 5* that $b = 2m$ and $h = 2r$, the surface area of the water can be calculated to be:

$$A = bh$$
$$A = (2m)(2r)$$
$$A = 4mr$$
$$A = 4\left(\frac{r}{\sin\theta}\right)(r)$$
$$A = \frac{4r^2}{\sin\theta}$$

It is important to note that similar limitations apply to this formula as to the formula for the cylindrical container.

I found this topic very stimulating, because it is something that I can see for myself by tipping a can of water on its edge. I am always interested in new ways to apply mathematics to the real world, because often, interesting patterns can be drawn from the data. Finding new ways to connect trigonometry to such an everyday thing as water in a can is fascinating to me, and I will continue to look for potential opportunities for this focus area to be applied.

Bibliography

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Surface area

Criterion	Marks	Comments
A	3	Concise and complete. Contains an aim, rationale and an implicit conclusion.
B	2	Errors with Integration notation (pg 3), approximation sign and the table on Pg 5 is not really helpful.
C	3	There is evidence of significant personal engagement, but it lacked applications to real life.
D	2	There is evidence of meaningful reflection especially when talking about assumptions made, thickness of glass and discussing the orientation of tilts. Student could have discussed the angle of tilt at which maximum or minimum areas occur.
E HL / SL	5 / 6	At level commensurate with the course, where the student understood the equation of an ellipse and applied integration to find area of surface. It lacks rigour and sophistication and hence does not reach top achievement level for HL.
Total HL	15	
Total SL	16	