2014 P1 TZ1

3. *[Maximum mark: 5]*

Consider $a = \log_2 3 \times \log_3 4 \times \log_4 5 \times \ldots \times \log_{31} 32$. Given that $a \in \mathbb{Z}$, find the value of a.

13. *[Maximum mark: 17]*

A geometric sequence $\{u_n\}$, with complex terms, is defined by $u_{n+1} = (1+\mathrm{i})u_n$ and $u_1 = 3$.

(a) Find the fourth term of the sequence, giving your answer in the form $x + y\mathrm{i}$, x, $y \in \mathbb{R}$. *[3]*

(b) Find the sum of the first 20 terms of $\{u_n\}$, giving your answer in the form $a \times (1 + 2^m)$ where $a \in \mathbb{C}$ and $m \in \mathbb{Z}$ are to be determined. *[4]*

A second sequence $\{v_n\}$ is defined by $v_n = u_n u_{n+k}$, $k \in \mathbb{N}$.

(c) (i) Show that $\{v_n\}$ is a geometric sequence.

(ii) State the first term.

(iii) Show that the common ratio is independent of k. *[5]*

A third sequence $\{w_n\}$ is defined by $w_n = |u_n - u_{n+1}|$.

(d) (i) Show that $\{w_n\}$ is a geometric sequence.

(ii) State the geometrical significance of this result with reference to points on the complex plane. *[5]*

2014 P1 TZ2

7. *[Maximum mark: 7]*

Consider the complex numbers $u = 2 + 3\mathrm{i}$ and $v = 3 + 2\mathrm{i}$.

(a) Given that $\frac{1}{u} + \frac{1}{v} = \frac{10}{w}$, express w in the form $a + b\mathrm{i}$, a, $b \in \mathbb{R}$. *[4]*

(b) Find w^* and express it in the form $re^{\mathrm{i}\theta}$. *[3]*

2014 P2 TZ1

4. *[Maximum mark: 6]*

A system of equations is given below.

$$x + 2y - z = 2$$
$$2x + y + z = 1$$
$$-x + 4y + az = 4$$

(a) Find the value of a so that the system does not have a unique solution. *[4]*

(b) Show that the system has a solution for any value of a. *[2]*

7. *[Maximum mark: 8]*

Prove, by mathematical induction, that $7^{8n+3} + 2$, $n \in \mathbb{N}$, is divisible by 5.

2014 P2 TZ2

1. *[Maximum mark: 6]*

(a) (i) Find the sum of all integers, between 10 and 200, which are divisible by 7.

(ii) Express the above sum using sigma notation. *[4]*

An arithmetic sequence has first term 1000 and common difference of -6. The sum of the first n terms of this sequence is negative.

(b) Find the least value of n. *[2]*

5. *[Maximum mark: 6]*

Find the coefficient of x^{-2} in the expansion of $(x-1)^3\left(\frac{1}{x} + 2x\right)^6$.

2015 P1 TZ2

7. [Maximum mark: 9]

(a) Find three distinct roots of the equation $8z^3 + 27 = 0$, $z \in \mathbb{C}$ giving your answers in modulus-argument form. [6]

The roots are represented by the vertices of a triangle in an Argand diagram.

(b) Show that the area of the triangle is $\frac{27\sqrt{3}}{16}$. [3]

9. [Maximum mark: 8]

(a) State the set of values of a for which the function $x \mapsto \log_a x$ exists, for all $x \in \mathbb{R}^+$. [2]

(b) Given that $\log_x y = 4\log_y x$, find all the possible expressions of y as a function of x. [6]

Markschemes

2014 P1 TZ1

3. $\frac{\log 3}{\log 2} \times \frac{\log 4}{\log 3} \times \ldots \times \frac{\log 32}{\log 31}$ ***M1A1***

$= \frac{\log 32}{\log 2}$ ***A1***

$= \frac{5\log 2}{\log 2}$ ***(M1)***

$= 5$ ***A1***

hence $a = 5$

[5 marks]

Note: Accept the above if done in a specific base *eg* $\log_2 x$.

13. (a) $r = 1 + \mathrm{i}$ *(A1)*

$u_4 = 3(1+\mathrm{i})^3$ *M1*

$= -6 + 6\mathrm{i}$ *A1*

[3 marks]

(b) $S_{20} = \dfrac{3\left((1+\mathrm{i})^{20} - 1\right)}{\mathrm{i}}$ *(M1)*

$= \dfrac{3\left((2i)^{10} - 1\right)}{\mathrm{i}}$ *(M1)*

Note: Only one of the two *M1*s can be implied. Other algebraic methods may be seen.

$= \dfrac{3\left(-2^{10} - 1\right)}{\mathrm{i}}$ *(A1)*

$= 3\mathrm{i}\left(2^{10} + 1\right)$ *A1*

[4 marks]

(c) (i) **METHOD 1**

$v_n = \left(3(1+\mathrm{i})^{n-1}\right)\left(3(1+\mathrm{i})^{n-1+k}\right)$ *M1*

$9(1+\mathrm{i})^k (1+i)^{2n-2}$ *A1*

$= 9(1+\mathrm{i})^k ((1+i)^2)^{n-1} \left(= 9(1+\mathrm{i})^k (2\mathrm{i})^{n-1}\right)$

this is the general term of a geometrical sequence *R1AG*

Notes: Do not accept the statement that the product of terms in a geometric sequence is also geometric unless justified further. If the final expression for v_n is $9(1+\mathrm{i})^k (1+i)^{2n-2}$ award *M1A1R0*.

METHOD 2

$\dfrac{v_{n+1}}{v_n} = \dfrac{u_{n+1}u_{n+k+1}}{u_n u_{n+k}}$ *M1*

$= (1+\mathrm{i})(1+\mathrm{i})$ *A1*

this is a constant, hence sequence is geometric *R1AG*

Note: Do not allow methods that do not consider the general term.

(ii) $9(1+\mathrm{i})^k$ *A1*

(iii) common ratio is $(1+i)^2 (= 2i)$ (which is independent of k) *A1*

[5 marks]

continued …

Question 13 continued

(d) (i) **METHOD 1**

$w_n = \left|3(1+i)^{n-1} - 3(1+\mathrm{i})^n\right|$ *M1*

$= 3\left|1+i\right|^{n-1}\left|1-(1+\mathrm{i})\right|$ *M1*

$= 3\left|1+i\right|^{n-1}$ *A1*

$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$

this is the general term for a geometric sequence *R1AG*

METHOD 2

$w_n = \left|u_n - (1+\mathrm{i})u_n\right|$ *M1*

$= \left|u_n\right|\left|-\mathrm{i}\right|$

$= \left|u_n\right|$ *A1*

$= \left|3(1+\mathrm{i})^{n-1}\right|$

$= 3\left|(1+\mathrm{i})\right|^{n-1}$ *A1*

$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$

this is the general term for a geometric sequence *R1AG*

Note: Do not allow methods that do not consider the general term.

(ii) distance between successive points representing u_n in the complex plane forms a geometric sequence *R1*

Note: Various possibilities but must mention distance between successive points.

[5 marks]

Total [17 marks]

7. (a) **METHOD 1**

$$\frac{1}{2+3i}+\frac{1}{3+2i}=\frac{2-3i}{4+9}+\frac{3-2i}{9+4}$$ ***M1A1***

$$\frac{10}{w}=\frac{5-5i}{13}$$ ***A1***

$$w=\frac{130}{5-5i}$$

$$=\frac{130\times5\times(1+i)}{50}$$

$$w=13+13i$$ ***A1***

[4 marks]

METHOD 2

$$\frac{1}{2+3i}+\frac{1}{3+2i}=\frac{3+2i+2+3i}{(2+3i)(3+2i)}$$ ***M1A1***

$$\frac{10}{w}=\frac{5+5i}{13i}$$ ***A1***

$$\frac{w}{10}=\frac{13i}{5+5i}$$

$$w=\frac{130i}{(5+5i)}\times\frac{(5-5i)}{(5-5i)}$$

$$=\frac{650+650i}{50}$$

$$=13+13i$$ ***A1***

[4 marks]

(b) $w^{*}=13-13i$ ***A1***

$$z=\sqrt{338}e^{-\frac{\pi}{4}i}\left(=13\sqrt{2}\,e^{-\frac{\pi}{4}i}\right)$$ ***A1A1***

> **Note:** Accept $\theta=\frac{7\pi}{4}$.
> Do not accept answers for θ given in degrees.

[3 marks]

Total [7 marks]

2014 P2 TZ1

4. (a) $\begin{cases} x+2y-z=2 \\ 2x+y+z=1 \\ -x+4y+az=4 \end{cases}$

$\rightarrow \begin{cases} x+2y-z=2 \\ -3y+3z=-3 \\ 6y+(a-1)z=6 \end{cases}$ ***M1A1***

$\rightarrow \begin{cases} x+2y-z=2 \\ -3y+3z=-3 \\ (a+5)z=0 \end{cases}$ ***A1***

(or equivalent)

if not a unique solution then $a=-5$ ***A1***

> **Note:** The first ***M1*** is for attempting to eliminate a variable, the first ***A1*** for obtaining two expression in just two variables (plus a), and the second ***A1*** for obtaining an expression in just a and one other variable

[4 marks]

(b) if $a=-5$ there are an infinite number of solutions as last equation always true ***R1***

and if $a \neq -5$ there is a unique solution ***R1***

hence always a solution ***AG***

[2 marks]

Total [6 marks]

7. if $n=0$

$7^3+2=345$ which is divisible by 5, hence true for $n=0$ *A1*

Note: Award ***A0*** for using $n=1$ but do not penalize further in question.

assume true for $n=k$ *M1*

Note: Only award the ***M1*** if truth is assumed.

so $7^{8k+3}+2=5p,\ p\in$ • *A1*

if $n=k+1$

$7^{8(k+1)+3}+2$ *M1*

$=7^8 7^{8k+3}+2$ *M1*

$=7^8(5p-2)+2$ *A1*

$=7^8.5p-2.7^8+2$

$=7^8.5p-11529600$

$=5\left(7^8 p-2305920\right)$ *A1*

hence if true for $n=k$, then also true for $n=k+1$. Since true for $n=0$, then true for all $n\in$ • *R1*

[8 marks]

Note: Only award the ***R1*** if the first two ***M1***s have been awarded.

2014 P2 TZ2

1. (a) (i) $n = 27$ *(A1)*

METHOD 1

$S_{27} = \frac{14+196}{2} \times 27$ *(M1)*

$= 2835$ ***A1***

METHOD 2

$S_{27} = \frac{27}{2}(2\times 14 + 26\times 7)$ *(M1)*

$= 2835$ ***A1***

METHOD 3

$S_{27} = \sum_{n=1}^{27} 7 + 7n$ *(M1)*

$= 2835$ ***A1***

(ii) $\sum_{n=1}^{27}(7+7n)$ or equivalent ***A1***

Note: Accept $\sum_{n=2}^{28} 7n$

[4 marks]

(b) $\frac{n}{2}\left(2000 - 6(n-1)\right) < 0$ *(M1)*

$n > 334.333$

$n = 335$ ***A1***

Note: Accept working with equalities.

[2 marks]

Total [6 marks]

5. expanding $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ *A1*

expanding $\left(\frac{1}{x} + 2x\right)^6$ gives

$64x^6 + 192x^4 + 240x^2 + \frac{60}{x^2} + \frac{12}{x^4} + \frac{1}{x^6} + 160$ ***(M1)A1A1***

Note:	Award ***(M1)*** for an attempt at expanding using binomial. Award ***A1*** for $\frac{60}{x^2}$. Award ***A1*** for $\frac{12}{x^4}$.

$\frac{60}{x^2} \times -1 + \frac{12}{x^4} \times -3x^2$ ***(M1)***

Note:	Award ***(M1)*** only if both terms are considered.

therefore coefficient x^{-2} is -96 ***A1***

Note:	Accept $-96x^{-2}$

Note:	Award full marks if working with the required terms only without giving the entire expansion.

[6 marks]

2015 P1 TZ2

7. (a) **METHOD 1**

$$z^3 = -\frac{27}{8} = \frac{27}{8}(\cos\pi + \mathrm{i}\sin\pi)$$ ***M1(A1)***

$$= \frac{27}{8}(\cos(\pi + 2n\pi) + \mathrm{i}\sin(\pi + 2n\pi))$$ ***(A1)***

$$z = \frac{3}{2}\left(\cos\left(\frac{\pi + 2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi + 2n\pi}{3}\right)\right)$$ ***M1***

$$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right),$$

$$z_2 = \frac{3}{2}(\cos\pi + \mathrm{i}\sin\pi),$$

$$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right).$$ ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $\left(re^{\mathrm{i}\theta}\right)$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

METHOD 2

$8z^3 + 27 = 0$

$\Rightarrow z = -\frac{3}{2}$ so $(2z + 3)$ is a factor

Attempt to use long division or factor theorem: ***M1***

$\Rightarrow 8z^3 + 27 \equiv (2z + 3)(4z^2 - 6z + 9)$

$\Rightarrow 4z^2 - 6z + 9 = 0$ ***A1***

Attempt to solve quadratic: ***M1***

$z = \frac{3 \pm 3\sqrt{3}\,\mathrm{i}}{4}$ ***A1***

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$,

$z_2 = \frac{3}{2}(\cos\pi + \mathrm{i}\sin\pi)$,

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right)$. ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $(re^{i\theta})$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

METHOD 3

$8z^3+27=0$

Substitute $z=x+\mathrm{i}y$ ***M1***

$8\left(x^3+3\mathrm{i}x^2y-3xy^2-\mathrm{i}y^3\right)+27=0$

$\Rightarrow 8x^3-24xy^2+27=0$ and $24x^2y-8y^3=0$ ***A1***

Attempt to solve simultaneously: ***M1***

$8y\left(3x^2-y^2\right)=0$

$y=0, y=x\sqrt{3}, y=-x\sqrt{3}$

$\Rightarrow\left(x=-\frac{3}{2}, y=0\right), x=\frac{3}{4}, y=\pm\frac{3\sqrt{3}}{4}$ ***A1***

$z_1=\frac{3}{2}\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right),$

$z_2=\frac{3}{2}(\cos\pi+\mathrm{i}\sin\pi),$

$z_3=\frac{3}{2}\left(\cos\frac{5\pi}{3}+\mathrm{i}\sin\frac{5\pi}{3}\right).$ ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $\left(re^{\mathrm{i}\theta}\right)$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

(b) **EITHER**

Valid attempt to use area $= 3\left(\frac{1}{2}ab\sin C\right)$ ***M1***

$= 3 \times \frac{1}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{\sqrt{3}}{2}$ ***A1A1***

Note: Award ***A1*** for correct sides, ***A1*** for correct $\sin C$.

OR

Valid attempt to use area $= \frac{1}{2}$ base $\times$ height ***M1***

area $= \frac{1}{2} \times \left(\frac{3}{4} + \frac{3}{2}\right) \times \frac{6\sqrt{3}}{4}$ ***A1A1***

Note: ***A1*** for correct height, ***A1*** for correct base.

THEN

$= \frac{27\sqrt{3}}{16}$ ***AG***

[3 marks]

Total [9 marks]

9. (a) $a > 0$ *A1*

$a \neq 1$ *A1*

[2 marks]

(b) **METHOD 1**

$\log_x y = \frac{\ln y}{\ln x}$ and $\log_y x = \frac{\ln x}{\ln y}$ *M1A1*

Note: Use of any base is permissible here, not just "e".

$\left(\frac{\ln y}{\ln x}\right)^2 = 4$ *A1*

$\ln y = \pm 2 \ln x$ *A1*

$y = x^2$ or $\frac{1}{x^2}$ *A1A1*

METHOD 2

$\log_y x = \frac{\log_x x}{\log_x y} = \frac{1}{\log_x y}$ *M1A1*

$(\log_x y)^2 = 4$ *A1*

$\log_x y = \pm 2$ *A1*

$y = x^2$ or $y = \frac{1}{x^2}$ *A1A1*

Note: The final two ***A*** marks are independent of the one coming before.

[6 marks]

Total [8 marks]