HL Math Review Topic 1: Algebra

Sequences and Series

1. If the sequence $15, 5x-4, \frac{x}{5}$ is arithmetic, what is the exact value of x?

$5x-4-15 = \frac{x}{5} - (5x-4)$

$5x-19 = -\frac{24x}{5} + 4$

$\frac{49x}{5} = 23$

$49x = 115$

$x = \frac{115}{49}$

2. Write each series in sigma notation.

a. $-5-2+1+\ldots+124$

$+3$ $+3$

$124 = -5+3(n-1)$

$129 = 3(n-1)$

$43 = n-1$

$44 = n$

$$\boxed{\sum_{n=1}^{44}\left(-5+3(n-1)\right)}$$

b. $\frac{7}{18}+\frac{14}{19}+\frac{21}{20}+\ldots+\frac{63}{26}$

$$\boxed{\sum_{n=1}^{9} \frac{7n}{18+(n-1)}}$$

c. $2+\frac{5}{2}+\frac{25}{8}+\frac{125}{32}+\frac{625}{128}$

$$\boxed{\sum_{n=0}^{4} 2\left(\frac{5}{4}\right)^n}$$

3. The first term of an infinite geometric sequence is 98, while the third term is 32. There are two possible sequences. Find the sum of each sequence.

$32 = 98r^2$

$\frac{16}{49} = r^2$

$r = \pm\frac{4}{7}$

Sum 1: $\frac{98}{1-\frac{4}{7}} = 98 \cdot \frac{7}{3} = \frac{686}{3} \approx \boxed{229}$

Sum 2: $\frac{98}{1+\frac{4}{7}} = 98 \cdot \frac{7}{11} = \frac{686}{11} \approx \boxed{62.4}$

Mathematical Induction

How every induction proof goes:

Step 1: State Proposition P_n.

Step 2: Show that the proposition P_a is true.

Step 3: Assume that P_k is true and show that thus P_{k+1} is true.

Step 4: Make the conclusion:

"Hence if P_k is true, then P_{k+1} is true.

Since P_a is true, then P_n is true for all $n \in \mathbb{Z}, n \geq a$."

4. Consider the sum $S_n = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{n\cdot(n+1)}$

a. Make a conjecture about the sum.

b. Use mathematical induction to prove your conjecture.

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2}+\frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

$S_3 = \frac{1}{2}+\frac{1}{6}+\frac{1}{12} = \frac{9}{12} = \frac{3}{4}$

$S_4 = \frac{9}{12}+\frac{1}{20} = \frac{48}{60} = \frac{4}{5}$

a. $S_n = \frac{n}{n+1}$

1 b. P_n is $S_n = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)} = \frac{n}{n+1}$

2 P_1: $\frac{1}{1+1} = \frac{1}{2}$ ✓

3 Assume P_k is true.

Then $S_k = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)} = \frac{k}{k+1}$

Consider P_{k+1}.

$S_{k+1} = S_k + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{k^2+2k+1}{(k+1)(k+2)}$

$= \frac{\cancel{(k+1)}(k+1)}{\cancel{(k+1)}(k+2)}$

$= \frac{k+1}{k+2}$ ✓

4 Hence if P_k is true, then P_{k+1} is true. Since P_1 is true, then P_n is true for all $n \in \mathbb{Z}^+$.

Complex Numbers

Cartesian Form: $z = a + bi$	Polar Form: $z = |z| cis\theta$	Euler's Form: $z = |z| e^{i\theta}$

5. Evaluate a. $4e^{i\left(\frac{-2\pi}{3}\right)}$

$4\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$

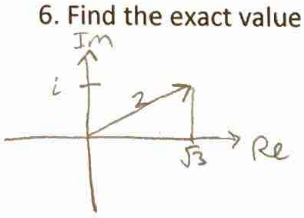
$4\left(-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right) = \boxed{-2 - 2i\sqrt{3}}$

b. $i^{-i} = \left(e^{i\frac{\pi}{2}}\right)^{-i} = e^{-i^2 \cdot \frac{\pi}{2}} = \boxed{e^{\frac{\pi}{2}}}$

DeMoivre's Theorem: $\left(|z| cis\theta\right)^n$ = $|z|^n cis\, n\theta$ for all rational n.

$z^n = \left(|z| e^{i\theta}\right)^n = |z|^n e^{in\theta}$

6. Find the exact value of $\left(\sqrt{3} + i\right)^8$.



$\left(2 cis\frac{\pi}{6}\right)^8 = 2^8 cis\, 8 \cdot \frac{\pi}{6} = 256\, cis\frac{4\pi}{3}$

$= 256\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \boxed{-128 - 128\sqrt{3}\, i}$

The **nth roots of a complex number** z are the solutions of $z^n = c$.

- There are exactly <u>n</u> n^{th} roots of c.
- If c is real, then the complex roots must occur in <u>conjugate</u> pairs.
- The roots of z^n will all have the same modulus which is <u>$|z|^{1/n}$</u>.
- On an Argand diagram, the roots all lie on a circle with radius $r =$ <u>$|z|^{1/n}$</u>, and the roots are equally spaced around that circle.

4a. Write 16 in polar form.

$16 = 16\, cis\, 2\pi$

b. Hence, find the 4^{th} roots of 16.

$16 = 16\, cis(2\pi n)$

$\left(16\, cis(2\pi n)\right)^{\frac{1}{4}} = 16^{\frac{1}{4}} \cdot cis\frac{2\pi n}{4}$

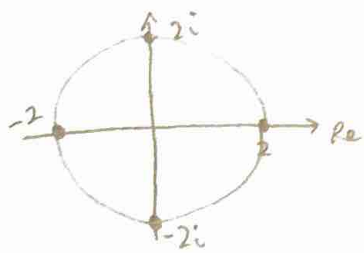
$= 2\, cis\frac{\pi}{2}n$

$n = 0 \rightarrow 2$

$n = 1 \rightarrow 2i$

$n = 2 \rightarrow -2$

$n = 3 \rightarrow -2i$



Linear Systems

7. Describe the possible solutions for the system and when they occur.

a. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 14 & 10-2a \end{bmatrix}$ unique solution

b. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 10-2a \end{bmatrix}$ $a = 5 \rightarrow$ infinite sols. $a \neq 5 \rightarrow$ no solution

c. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 7-a & 12 \end{bmatrix}$ $a = 7 \rightarrow$ no sol. $a \neq 7 \rightarrow$ unique sol.

d. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 7-a & 0 \end{bmatrix}$ $a = 7 \rightarrow$ infinite sols $a \neq 7 \rightarrow$ unique sol.

The Binomial Theorem

8. Find a if the coefficient of x^7 in the expansion of $(3x + a)^{12}$ is -228096.

$\binom{12}{5}(3x)^7(a)^5 = -228096x^7$

$792 \cdot 2187\, a^5 = -228096$

$a^5 = \frac{-32}{243} \rightarrow \boxed{a = \frac{-2}{3}}$