Trig Substitution Practice

Do these problems only: 1 – 6, 10 – 12, 13 – 15

In Exercises 1–4, state the trigonometric substitution you would use to find the integral. Do not integrate.

1. $\int (9 + x^2)^{-2}\,dx$

2. $\int \sqrt{4 - x^2}\,dx$

3. $\int \frac{x^2}{\sqrt{16 - x^2}}\,dx$

4. $\int x^2(x^2 - 25)^{3/2}\,dx$

In Exercises 5–8, find the indefinite integral using the substitution $x = 4\sin\theta$.

5. $\int \frac{1}{(16 - x^2)^{3/2}}\,dx$

6. $\int \frac{4}{x^2\sqrt{16 - x^2}}\,dx$

7. $\int \frac{\sqrt{16 - x^2}}{x}\,dx$

8. $\int \frac{x^2}{\sqrt{16 - x^2}}\,dx$

In Exercises 9–12, find the indefinite integral using the substitution $x = 5\sec\theta$.

9. $\int \frac{1}{\sqrt{x^2 - 25}}\,dx$

10. $\int \frac{\sqrt{x^2 - 25}}{x}\,dx$

11. $\int x^3\sqrt{x^2 - 25}\,dx$

12. $\int \frac{x^3}{\sqrt{x^2 - 25}}\,dx$

In Exercises 13–16, find the indefinite integral using the substitution $x = \tan\theta$.

13. $\int x\sqrt{1 + x^2}\,dx$

14. $\int \frac{9x^3}{\sqrt{1 + x^2}}\,dx$

15. $\int \frac{1}{(1 + x^2)^2}\,dx$

16. $\int \frac{x^2}{(1 + x^2)^2}\,dx$

Answers

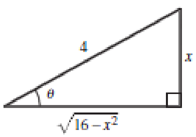
1. Use $x = 3\tan\theta$

2. Use $x = 2\sin\theta$

3. Use $x = 4\sin\theta$

4. Use $x = 5\sec\theta$

5. Let $x = 4\sin\theta, dx = 4\cos\theta\, d\theta, \sqrt{16 - x^2} = 4\cos\theta.$



$$\int \frac{1}{(16 - x^2)^{3/2}}\,dx = \int \frac{4\cos\theta}{(4\cos\theta)^3}\,d\theta = \frac{1}{16}\int \sec^2\theta\, d\theta = \frac{1}{16}\tan\theta + C = \frac{1}{16}\frac{x}{\sqrt{16 - x^2}} + C$$

6. Same substitution as in Exercise 5.

$$\int \frac{4}{x^2\sqrt{16 - x^2}}\,dx = 4\int \frac{4\cos\theta}{(4\sin\theta)^2(4\cos\theta)}\,d\theta = \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C = -\frac{1}{4}\frac{\sqrt{16 - x^2}}{x} + C = \frac{-\sqrt{16 - x^2}}{4x} + C$$

7. Same substitution as in Exercise 5

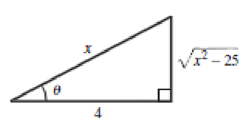
$$\begin{aligned}
\int \frac{\sqrt{16 - x^2}}{x}\,dx &= \int \frac{4\cos\theta}{4\sin\theta}4\cos\theta\, d\theta \\
&= 4\int \frac{\cos^2\theta}{\sin\theta}\,d\theta \\
&= 4\int \frac{1 - \sin^2\theta}{\sin\theta}\,d\theta \\
&= 4\int (\csc\theta - \sin\theta)\,d\theta \\
&= -4\ln|\csc\theta + \cot\theta| + 4\cos\theta + C \\
&= -4\ln\left|\frac{4}{x} + \frac{\sqrt{16 - x^2}}{x}\right| + 4\frac{\sqrt{16 - x^2}}{4} + C \\
&= -4\ln\left|\frac{4 + \sqrt{16 - x^2}}{x}\right| + \sqrt{16 - x^2} + C \\
&= 4\ln\left|\frac{4 - \sqrt{16 - x^2}}{x}\right| + \sqrt{16 - x^2} + C
\end{aligned}$$

8. Same substitution as in Exercise 5.

$$\begin{aligned}
\int \frac{x^2}{\sqrt{16 - x^2}}\,dx &= \int \frac{(4\sin\theta)^2}{4\cos\theta}4\cos\theta\, d\theta \\
&= 16\int \sin^2\theta\, d\theta \\
&= 8\int (1 - \cos 2\theta)\,d\theta \\
&= 8\left(\theta - \frac{\sin 2\theta}{2}\right) + C \\
&= 8(\theta - \sin\theta\cos\theta) + C \\
&= 8\left(\arcsin\frac{x}{4} - \frac{x}{4}\frac{\sqrt{16 - x^2}}{4}\right) + C \\
&= 8\arcsin\frac{x}{4} - \frac{x\sqrt{16 - x^2}}{2} + C
\end{aligned}$$

9. Let $x = 5\sec\theta, dx = 5\sec\theta\tan\theta\, d\theta,$

$\sqrt{x^2 - 25} = 5\tan\theta$



$$\begin{aligned}
\int \frac{1}{\sqrt{x^2 - 25}}\,dx &= \int \frac{5\sec\theta\tan\theta}{5\tan\theta}\,d\theta \\
&= \int \sec\theta\, d\theta \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{x}{5} + \frac{\sqrt{x^2 - 25}}{5}\right| + C \\
&= \ln\left|x + \sqrt{x^2 - 25}\right| + C
\end{aligned}$$

10. Same substitution as in Exercise 9

$$\int \frac{\sqrt{x^2 - 25}}{x}\,dx = \int \frac{5\tan\theta}{5\sec\theta}\,5\sec\theta\tan\theta\,d\theta$$
$$= 5\int \tan^2\theta\,d\theta$$
$$= 5\int \left(\sec^2\theta - 1\right)d\theta$$
$$= 5(\tan\theta - \theta) + C$$
$$= 5\left(\frac{\sqrt{x^2 - 25}}{5} - \operatorname{arcsec}\frac{x}{5}\right) + C$$
$$= \sqrt{x^2 - 25} - 5\operatorname{arcsec}\frac{x}{5} + C$$

$$\left[\textbf{Note: } \operatorname{arcsec}\left(\frac{x}{5}\right) = \arctan\left(\frac{\sqrt{x^2 - 25}}{5}\right)\right]$$

11. Same substitution as in Exercise 9

$$\int x^3\sqrt{x^2 - 25}\,dx = \int (5\sec\theta)^3(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta$$
$$= 3125\int \sec^4\theta\tan^2\theta\,d\theta$$
$$= 3125\int \left(1 + \tan^2\theta\right)\tan^2\theta\sec^2\theta\,d\theta$$
$$= 3125\int \left(\tan^2\theta + \tan^4\theta\right)\sec^2\theta\,d\theta$$
$$= 3125\left[\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5}\right] + C$$
$$= 3125\left[\frac{\left(x^2 - 25\right)^{3/2}}{125(3)} + \frac{\left(x^2 - 25\right)^{5/2}}{5^5(5)}\right] + C$$
$$= \frac{1}{15}\left(x^2 - 25\right)^{3/2}\left[125 + 3\left(x^2 - 25\right)\right] + C$$
$$= \frac{1}{15}\left(x^2 - 25\right)^{3/2}\left(50 + 3x^2\right) + C$$

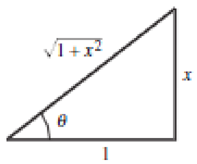
12. Same substitution as in Exercise 9

$$\int \frac{x^3}{\sqrt{x^2 - 25}}\,dx = \int \frac{(5\sec\theta)^3}{5\tan\theta}\,5\sec\theta\tan\theta\,d\theta$$
$$= 125\int \sec^4\theta\,d\theta$$
$$= 125\int \left(\tan^2\theta + 1\right)\sec^2\theta\,d\theta$$
$$= 125\left(\frac{\tan^3\theta}{3} + \tan\theta\right) + C$$
$$= \frac{125}{3}\frac{\left(x^2 - 25\right)^{3/2}}{125} + 125\frac{\sqrt{x^2 - 25}}{5} + C$$
$$= \frac{1}{3}\left(x^2 - 25\right)^{3/2} + 25\left(x^2 - 25\right)^{1/2} + C$$
$$= \frac{1}{3}\sqrt{x^2 - 25}\left(x^2 - 25 + 75\right) + C$$
$$= \frac{1}{3}\sqrt{x^2 - 25}\left(50 + x^2\right) + C$$

13. Let $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $\sqrt{1 + x^2} = \sec\theta$.

$$\int x\sqrt{1 + x^2}\, dx = \int \tan\theta(\sec\theta)\sec^2\theta\, d\theta = \frac{\sec^3\theta}{3} + C = \frac{1}{3}(1 + x^2)^{3/2} + C$$

Note: This integral could have been evaluated with the Power Rule.



14. Same substitution as in Exercise 13

$$\int \frac{9x^3}{\sqrt{1 + x^2}}\, dx = 9\int \frac{\tan^3\theta}{\sec\theta}\sec^2\theta\, d\theta = 9\int(\sec^2\theta - 1)\sec\theta\tan\theta\, d\theta = 9\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$

$$= 3\sec\theta(\sec^2\theta - 3) + C = 3\sqrt{1 + x^2}\left[(1 + x^2) - 3\right] + C = 3\sqrt{1 + x^2}(x^2 - 2) + C$$

15. Same substitution as in Exercise 13

$$\int \frac{1}{(1 + x^2)^2}\, dx = \int \frac{1}{\left(\sqrt{1 + x^2}\right)^4}\, dx = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$$

$$= \int \cos^2\theta\, d\theta = \frac{1}{2}\int(1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$

$$= \frac{1}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{2}\left[\arctan x + \left(\frac{x}{\sqrt{1 + x^2}}\right)\left(\frac{1}{\sqrt{1 + x^2}}\right)\right] + C$$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{1 + x^2}\right) + C$$

16. Same substitution as in Exercise 13

$$\int \frac{x^2}{(1 + x^2)^2}\, dx = \int \frac{x^2}{\left(\sqrt{1 + x^2}\right)^4}\, dx = \int \frac{\tan^2\theta\sec^2\theta\, d\theta}{\sec^4\theta} = \int \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int(1 - \cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] = \frac{1}{2}\left[\theta - \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{2}\left[\arctan x - \left(\frac{x}{\sqrt{1 + x^2}}\right)\left(\frac{1}{\sqrt{1 + x^2}}\right)\right] + C = \frac{1}{2}\left(\arctan x - \frac{x}{1 + x^2}\right) + C$$