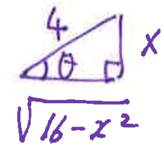
Warm-up

1) Integrate using $\underline{x = 4\sin\theta}$

$$\int \frac{\sqrt{16-x^2}}{x}\,dx$$

$\sin\theta = \frac{x}{4}$



$dx = 4\cos\theta\, d\theta$

$4\cos\theta = \sqrt{16-x^2}$

$$= \int \frac{\cancel{4}\cos\theta}{\cancel{4}\sin\theta}\, 4\cos\theta\, d\theta$$

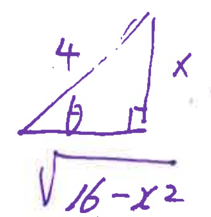
$$= 4\int \frac{\cos^2\theta}{\sin\theta}\,d\theta = 4\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = 4\int(\csc\theta - \sin\theta)\,d\theta$$

$$= 4\ln\left|\csc\theta - \cot\theta\right| + 4\cos\theta + C$$

$\csc\theta = \frac{4}{x}$

$\cot\theta = \frac{\sqrt{16-x^2}}{x}$

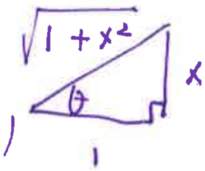
$\cos\theta = \frac{\sqrt{16-x^2}}{4}$



$$4\ln\left|\frac{4}{x} - \frac{\sqrt{16-x^2}}{x}\right| + \cancel{4}\frac{\sqrt{16-x^2}}{\cancel{4}} + C$$

2) Integrate using $x = \tan\theta$ $\quad dx = \sec^2\theta\,d\theta$

$$\int \frac{x^2}{(1+x^2)^2}\,dx.$$



$\sec\theta = \sqrt{1+x^2}$

$\sec^2\theta = 1+x^2$

$$= \int \frac{\tan^2\theta}{\sec^{\cancel{4}}\theta}\,\sec^{\cancel{2}}\theta\,d\theta \qquad \sec^2\theta$$

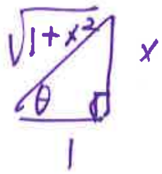
$$= \int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin^2\theta}{\cancel{\cos^2\theta}}\cdot\cancel{\cos^2\theta}\,d\theta = \int \sin^2\theta\,d\theta.$$

$$= \frac{1}{2}\int(1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}(\arctan x) - \frac{1}{2}\left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right) + C$$



$$= \frac{1}{2}\arctan x - \frac{x}{2(1+x^2)} + C.$$

$\tan\theta = x \Rightarrow \theta = \arctan x.$

$$\bullet \int \csc\theta \, d\theta \, \frac{(\csc\theta - \cot\theta)}{(\csc\theta - \cot\theta)} = \int \frac{\csc^2\theta - \csc\theta\cot\theta}{\csc\theta - \cot\theta} \, d\theta$$

$u = \csc\theta - \cot\theta$.

$du = \csc^2\theta - \csc\theta\cot\theta \, d\theta$

$$= \int \frac{du}{u} = \ln|\csc\theta - \cot\theta| + c$$

$$\bullet \int \underline{\sec\theta \, d\theta} = \frac{(\sec\theta + \tan\theta)}{(\sec\theta + \tan\theta)} = \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \, d\theta.$$

$$= \ln|\sec\theta + \tan\theta| + C$$