First Order Differential Equation WS #2.

#1. $p=2 \quad I(x)=e^{\int 2dx}=e^{2x}$

$e^{2x}\cdot\frac{dy}{dx}+2\cdot e^{2x}y=e^{x}$

$\Rightarrow \int\frac{d}{dx}\left[e^{2x}\cdot y\right]dx=\int e^{x}dx$

$y=e^{-2x}\cdot\left[e^{x}+c\right]=\boxed{e^{-x}+e^{-2x}c}$

#2. $p=-\frac{1}{2} \quad I(x)=e^{\int-\frac{1}{2}dx}=e^{-\frac{1}{2}x}$

$e^{-\frac{1}{2}x}\cdot\frac{dy}{dx}-\frac{1}{2}\cdot e^{-\frac{1}{2}x}\cdot y=\frac{1}{2}e^{-\frac{1}{2}x}\cdot e^{x/2}=\frac{1}{2}e^{0}$

$\Rightarrow \int\frac{d}{dx}e^{-\frac{1}{2}x}\cdot y\cdot dx=\int\frac{1}{2}dx \qquad \boxed{y=\frac{1}{2}x\cdot e^{x/2}+ce^{x/2}}$

#3. $p=\frac{3}{x} \quad I(x)=e^{\int\frac{3}{x}dx}=e^{3\ln x}=x^{3}$

$x^{3}\cdot\frac{dy}{dx}+3x^{2}y=\cancel{x^{3}}\cdot\frac{\sin x}{\cancel{x^{3}}} \qquad \left[\frac{dy}{dx}+\frac{3}{x}y=\frac{\sin x}{x^{3}}\right]$

$\Rightarrow \int\frac{d}{dx}\left(x^{3}\cdot y\right)dx=\int\sin x\,dx$

$y=\frac{1}{x^{3}}\int\sin x\,dx$

$\boxed{y=\frac{1}{x^{3}}\left[-\cos x+c\right.}$

#4. $x\frac{dy}{dx}+y=\sin x \quad\Rightarrow\quad \frac{dy}{dx}+\frac{1}{x}y=\frac{\sin x}{x}$

$\left(p=\frac{1}{x} \quad I(x)=e^{\int\frac{1}{x}dx}=e^{\ln x}=x\right)$

$\Rightarrow x\frac{dy}{dx}+y=\sin x. \quad\Rightarrow \int\frac{d}{dx}xy=\int\sin x\,dx$

$\boxed{y=\frac{1}{x}\left[-\cos x+c\right]}$

#5. $\frac{dy}{dx} + \frac{y}{x} = \frac{y}{x}\cdot dy$ $\left(p = \frac{1}{x} \quad I(x) = e^{\int \frac{1}{x}dx} = x \right)$

$\Rightarrow x\cdot\frac{dy}{dx} + y = y\,dy$

$\Rightarrow \int \frac{d(x\cdot y)}{dx} = \int y\,dy \Rightarrow \boxed{y = \frac{1}{x}\left[\frac{1}{2}y^2 + c\right]}$

#6. $\frac{dy}{dx} + \frac{4\cancel{(x-1)}^2}{\cancel{(x-1)}^3}y = \frac{x+1}{(x-1)^3}$ $\left(p = 4 \quad I(x) = e^{\int 4dx} = e^{4x} \right.$

$\Rightarrow \int e^{4x}\cdot\frac{dy}{dx} + 4\cdot e^{4x}\cdot y = \int e^{4x}\left(\frac{x+1}{(x-1)^3}\right)dx$

$y = \frac{1}{e^{4x}}\int e^{4x}\cdot\left(\frac{x+1}{(x-1)^3}\right)dx$ $\quad u = x-1 \quad du = dx$

$x = u+1$

$u+2 = x+1$

$= \frac{1}{e^{4x}}\int e^{4u+4}\left(\frac{u+2}{u^3}\right)du$

$4x = 4u+4$

$= \frac{1}{e^{4x}}\cdot e^4\left[\int e^{4u}\cdot\frac{u}{u^3}du + \int e^{4u}\cdot\frac{2}{u^3}du\right]$

$\frac{dy}{dx} + \left(\frac{4}{x-1}\right)y = \frac{x+1}{(x-1)^3}$ $\quad p = \frac{4}{x-1}$

#5 $\Rightarrow (x-1)^4 y' + 4(x-1)^3 y = (x-1)(x+1)$ $\quad I(x) = e^{\int\frac{4}{x-1}dx} = e^{\ln(x-1)^4} = (x-1)^4$

$\Rightarrow \int \frac{d(x-1)^4 y}{dx} = \int (x-1)(x+1)dx$

$= \int (x^2-1)dx$

$\boxed{y = \frac{1}{(x-1)^4}\left(\frac{1}{3}x^3 - x + c\right)}$

#6. $e^{2y} + 2(xe^{2y} - y)\frac{dy}{dx} = 0$ $\quad y' + py = Q$

$\Rightarrow e^{2y}\frac{dx}{dy} + 2xe^{2y} = 2y$ $\quad x' + px = Q$

$\Rightarrow \frac{dx}{dy} + 2x = \frac{2y}{e^{2y}}$ $\quad p = 2 \quad I(y) = e^{\int 2dy} = e^{2y}$

$\Rightarrow e^{2y}\cdot x' + e^{2y}(2x) = \cancel{e^{2y}}\cdot\frac{2y}{\cancel{e^{2y}}}$

$\Rightarrow \int d(e^{2y})(x) = \int 2y\,dy$ $\quad x = \frac{1}{e^{2y}}\left(y^2 + c\right)$

#7 $\dfrac{(x-2y)}{y} + \dfrac{y}{y}\dfrac{dx}{dy} = 0 \Rightarrow x' + \dfrac{1}{y}x = 2.$

$p = \dfrac{1}{y} \quad I(y) = e^{\int \frac{1}{y} dy}$

$= y.$

$\Rightarrow yx' + x = 2y$

$\int \dfrac{d(xy)}{dy} = \int 2y\, dy \Rightarrow \boxed{x = \dfrac{1}{y}(y^2 + c)}$