IB Math HL1 Double and Half Angle Identities

Warm up: (Let's do together)

Cooper Toy Company has designed a new toy that uses a spring that follows a sinusoidal curve after you wind it up and start it. At $t = 4$ seconds, the end of the spring is at its highest point, 18 cm above the ground. Five seconds later, the spring is at its lowest point, which is 6 cm above the ground.

a. Find an equation that will determine the height, $h(t)$, of the spring at any time t.

$h(t) = 6\cos\left[\frac{\pi}{5}(t-4)\right] + 12.$

$h(t) = A\cos[B(x-h)] + k.$

Amplitude: $\frac{18-6}{2} = 6$ $\quad h = 4$

Axis: k: $\frac{18+6}{2} = 12.$

B: $\frac{2\pi}{\text{period}} = \frac{2\pi}{10} = \frac{\pi}{5}$

b. Find the height of the spring after 15 seconds.

$h(15) = 6\cos\left[\frac{\pi}{5}(15-4)\right] + 12 \simeq 16.8$ cm

c. Find the first time the height of the spring reaches 12 cm above the ground.

$12 = 6\cos\left[\frac{\pi}{5}(\boxed{t}-4)\right] + 12.$

$\boxed{t \approx 6.5 \text{ sec}}$

Double and Half Angle Identities (refer the Trig identities sheet)

1. Rewrite $\sin 2\theta$ using $\sin(x+y) = \sin x\cos y + \sin y \cos x$.

$\boxed{\sin 2\theta} = \sin(\theta+\theta) = \sin\theta\cdot\cos\theta + \sin\theta\cdot\cos\theta = \boxed{2\sin\theta\cos\theta}$

2. Rewrite $\cos 2\theta$ using $\cos(x+y) = \cos x\cos y - \sin x \sin y$.

$\boxed{\cos 2\theta} = \cos(\theta+\theta) = \cos\theta\cdot\cos\theta - \sin\theta\sin\theta$

$= \boxed{\cos^2\theta - \sin^2\theta} = \cos^2\theta - (1-\cos^2\theta)$

$= \boxed{2\cos^2\theta - 1}$

$\sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta = 1 - \sin^2\theta$

$= 1 - \sin^2\theta - \sin^2\theta$

$= \boxed{1 - 2\sin^2\theta}$

3. Rewrite $\tan 2\theta$ using $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$.

$\tan 2\theta = \tan(\theta+\theta) = \dfrac{\tan\theta + \tan\theta}{1 - \tan\theta\cdot\tan\theta} = \boxed{\dfrac{2\tan\theta}{1-\tan^2\theta}}$

4. Rewrite $\sin\left(\frac{\theta}{2}\right)$ using $\underline{\cos 2x = 1 - 2\sin^2 x}$. ⇒ Solve for Sin x.

$-1 \quad -1$

$$\frac{\cos 2x - 1}{-2} = \frac{-2\sin^2 x}{-2}. \Rightarrow \frac{1}{2}(1 - \cos 2x) = \sin^2 x.$$

$$\sin x = \pm\sqrt{\frac{1}{2}(1 - \cos 2x)}$$

Sub. ⇐ $x = \frac{\theta}{2}$, ⇔ $2x = \theta$

$$\boxed{\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}(1 - \cos\theta)}}$$

Practice) Rewrite $\cos\left(\frac{\theta}{2}\right)$ using $\cos 2x = 2\cos^2 x - 1$.

$$\cos 2x = 2\cos^2 x - 1$$

$+1 \quad +1$

$$\frac{2\cos^2 x}{2} = \frac{\cos 2x + 1}{2}$$

$$\sqrt{\cos^2 x} = \sqrt{\frac{1}{2}(\cos 2x + 1)}$$

$$\cos x = \pm\sqrt{\frac{1}{2}(\cos 2x + 1)}$$

$x = \frac{\theta}{2}$

$$\boxed{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1}{2}(\cos\theta + 1)}}$$

Double Angle Identities

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

$\cos 2x = 2\cos^2 x - 1$

$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

⇑ Given

⇔

Half Angle Identities

$\sin\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$

$\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$

$\tan\frac{x}{2} = \frac{1 - \cos x}{\sin x}$

$\tan\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}$

$\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}$

Not Given

Try This! (Examples)

1) Given $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$,

find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$

$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2}(1-\cos\theta)}$

① $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2}\left(\frac{5}{5} + \frac{4}{5}\right)} = \sqrt{\frac{9}{10}} = \boxed{\frac{3}{\sqrt{10}}}$ OR $\boxed{\frac{3\sqrt{10}}{10}}$

② $\cos\left(\frac{\theta}{2}\right) = +\sqrt{\frac{1-4/5}{2}} = \sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}}$ OR $\frac{10}{\sqrt{10}}$

③ $\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta} = \frac{1-\left(-\frac{4}{5}\right)}{3/5} = \boxed{3}$

More Examples)

Find the exact value. ⇙ If we assess these problems, the half of identities will be given.

1). $\tan\left(\frac{7\pi}{8}\right)$ ⇒ 2) $\sin\left(\frac{5\pi}{8}\right)$ 3). $\cos(195^\circ)$

⇒ $\tan\left(\frac{7\pi}{8}\right)$

$\left(\frac{7\pi}{8}\right) = \frac{1}{2}X$

$X = \frac{7\pi}{4}$

$\tan\left(\frac{X}{2}\right) = \frac{\sin X}{1+\cos X}$

$\tan\left(\frac{7\pi}{8}\right) = \frac{\sin\left(\frac{7\pi}{4}\right)}{1+\cos\left(\frac{7\pi}{4}\right)} = \frac{\left(\frac{-1}{\sqrt{2}}\right)\cdot\sqrt{2}}{\left(1+\frac{1}{\sqrt{2}}\right)\cdot\sqrt{2}}$

$= \boxed{\frac{-\sqrt{2}}{(\sqrt{2}+1)}}\frac{(\sqrt{2}-1)}{(\sqrt{2}-1)}$

$= \frac{-2+\sqrt{2}}{2+1} = \frac{-2+\sqrt{2}}{3}$

$\frac{5\pi}{8} = \frac{1}{2}X$

$X = \frac{5\pi}{4}$

$\sin\left(\frac{x}{2}\right) = +\sqrt{\frac{1-\cos x}{2}}$

$= +\sqrt{\frac{1-\cos\frac{5\pi}{4}}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\frac{\sqrt{2}}{\sqrt{2}}+\frac{\sqrt{2}}{2}}{2}}$

$= \sqrt{\frac{2+\sqrt{2}}{4}}$

$= \frac{\sqrt{2+\sqrt{2}}}{2}$