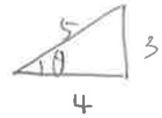
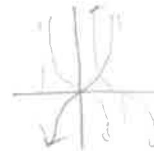
CIRCULAR FUNCTIONS AND TRIGONOMETRY

Question 1

Given that $0 \le \theta \le \frac{\pi}{2}$ and $\tan\theta = \frac{3}{4}$, find





(a) $\cos\theta$. $= \boxed{\frac{4}{5}}$

(b) $\sin 2\theta$. $= 2\sin\theta \cdot \cos\theta = (2)\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \boxed{\frac{24}{25}}$

(c) $\tan\left(\frac{\pi}{2} - \theta\right)$. $= \tan\left(-\theta + \frac{\pi}{2}\right) = \tan\left(-\left(\theta - \frac{\pi}{2}\right)\right) = -\tan\left(\theta - \frac{\pi}{2}\right) = \cot\theta$

$= \boxed{\frac{4}{3}}$.

Question 2

Given that $0 < \theta \le \frac{\pi}{2}$, arrange, in increasing order, $\sin\theta$, $\frac{1}{\sin\theta}$, $\sin^2\theta$.

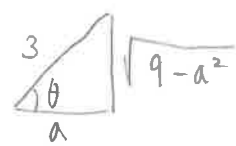
$\sin\theta$, $\csc\theta$, $(\sin\theta)^2$ $\Rightarrow$

$\Rightarrow \boxed{(\sin\theta)^2, \ \sin\theta, \ \frac{1}{\sin\theta}}$.

$0 \le \sin\theta \le 1$

Question 3

(a) If $0 < \theta < 90°$ and $\cos\theta = \frac{1}{3}a$, find $\sin\theta$.



$\sin\theta = \boxed{\frac{\sqrt{9-a^2}}{3}}$

(b) Express $\frac{\pi}{5}$ in degrees.

$\frac{\pi}{5} = \frac{180}{5} = 36°$

(c) Evaluate $\cos 300° \cos 30°$

$\cos 300° = \boxed{\frac{1}{2}}$, $\cos 30° = \boxed{\frac{\sqrt{3}}{2}}$

(d) Express in terms of $\tan\theta$, $\frac{\sin(\pi - \theta)}{\cos\left(\frac{\pi}{2} + \theta\right)} \cdot \tan\left(\frac{3\pi}{2} - \theta\right)$.

$= \frac{\cancel{\sin\theta}}{-\cancel{\sin\theta}} \cdot \frac{1}{\tan\theta} = \frac{-1}{\tan\theta}$

① $\sin(\pi - \theta)$

$= \sin(-[\theta + \pi])$

$= -\sin(\theta + \pi) = \sin\theta$.

② $\cos\left(\theta + \frac{\pi}{2}\right)$

$= -\sin\theta$.

$\tan\left(-\theta + \frac{3\pi}{2}\right)$

$= \tan\left[-\left(\theta - \frac{3\pi}{2}\right)\right]$

$= -\tan\left(\theta - \frac{3\pi}{2}\right) = \cot\theta$.

Question 4

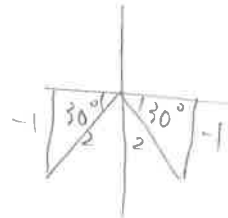
(a) Express $\frac{1}{\cos\theta - 1} - \frac{1}{\cos\theta + 1}$ in terms of $\sin\theta$.

(b) Solve $\frac{1}{\cos\theta - 1} - \frac{1}{\cos\theta + 1} = -8, 0° < \theta < 360°$

#4. a. $\frac{1}{\cos\theta - 1} - \frac{1}{\cos\theta + 1}$

$= \frac{(\cos\theta + 1) - (\cos\theta - 1)}{\cos^2\theta - 1} = \frac{2}{-\sin^2\theta}$

b. $\frac{\cancel{2}\,1}{-\sin^2\theta} = -\cancel{8}\,4.$ $(\theta \neq 180^\circ)$



$-\sin^2\theta = -\frac{1}{4} \Rightarrow \sin^2\theta = \frac{1}{4} \Rightarrow \sin\theta = \pm\frac{1}{2}$

$\theta = 210^\circ,\ \theta = 300^\circ$

#5. (a) $(\cos\theta - \sin\theta)^2$

$= \cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta$

$= 1 - (2)(\frac{1}{2}) = 0$

(b) $\sin^2\theta - \sin\theta = 0$

$0 \le \theta \le 2\pi$ $\quad \sin\theta(\sin\theta - 1) = 0$

$\sin\theta = 0 \qquad \sin\theta = 1$

$\theta = 0, 2\pi \qquad \theta = \frac{\pi}{2}$

$\sin^2\theta - \sin\theta = 2.$

$\sin^2\theta - \sin\theta - 2 = 0$

$\sin\theta \qquad -2$

$\sin\theta \qquad 1$

$\sin\theta = 2 \qquad \sin\theta = -1$

$\cancel{(\sin\theta - 2)}(\sin\theta + 1) = 0$

$\theta = \frac{3\pi}{2}$

Question 5

(a) Given that $\cos\theta\sin\theta = \frac{1}{2}$, evaluate $(\cos\theta - \sin\theta)^2$.

(b) Find all values of θ such that (i) $\sin^2\theta - \sin\theta = 0, 0 \le \theta \le 2\pi$.

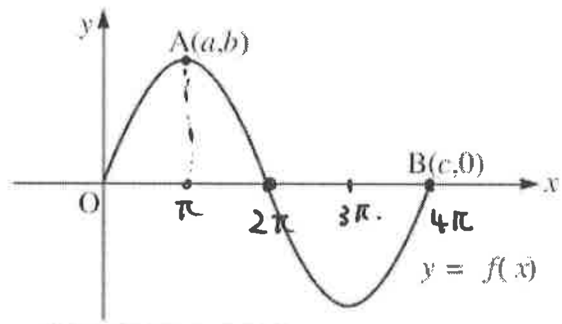
(ii) $\sin^2\theta - \sin\theta = 2, 0 \le \theta \le 2\pi$.

Question 6

The figure below shows the graph of $f(x) = 2\sin\left(\frac{x}{2}\right)$.

Amplitude: 2

Period: $\frac{2\pi}{\frac{1}{2}} = 4\pi$



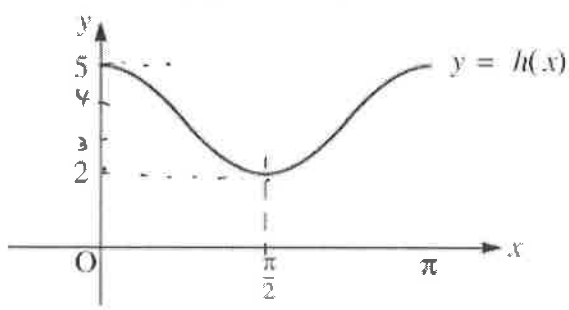
(a) Find a, b and c. $a = \pi$, $b = 2$, $c = 4\pi$

(b) Solve for x, where $f(x) = \sqrt{3}, 0 \le x \le c$.

$2\sin\left(\frac{x}{2}\right) = \sqrt{3} \Rightarrow \sin\left(\frac{x}{2}\right) = \frac{\sqrt{3}}{2} \Rightarrow \frac{x}{2} = \frac{\pi}{3}, \frac{2\pi}{3} \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$

Question 7

Consider the graph of the function $h(x) = a\cos(bx) + c$:



Find the values a, b and c.

$a = \frac{\max - \min}{2} = \frac{5-2}{2} = \boxed{\frac{3}{2}}$

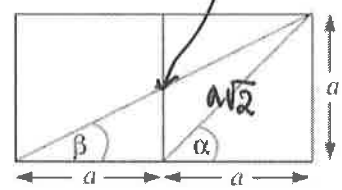
$c = \frac{\max + \min}{2} = \frac{5+2}{2} = \boxed{\frac{7}{2}}$

$b = \frac{2\pi}{\text{period}} = \frac{2\pi}{\pi} = \boxed{2.}$

Question 11

For the diagram shown alongside, the value of $\sin(\alpha - \beta) = \frac{1}{\sqrt{k}}$, where $k \in \mathbb{Z}^+$.

$\sqrt{a^2 + 4a^2} = \sqrt{5}\cdot a$



Find the value of k.

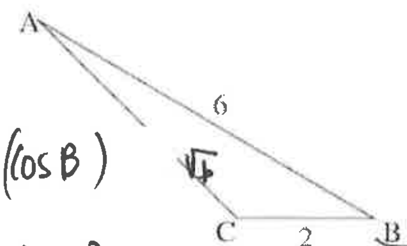
$$\sin(\alpha-\beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$$

$$= \frac{a}{a\sqrt{2}} \cdot \frac{2a}{\sqrt{5}\cdot a} - \frac{a}{\sqrt{2}\cdot a} \cdot \frac{a}{\sqrt{5}\cdot a} = \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{5}}$$

$$= \frac{2}{\sqrt{10}} - \frac{1}{\sqrt{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

Question 15

In the diagram below, $\cos B = \frac{3}{8}$, AB = 6 and BC = 2, AC = $\sqrt{b}$, find b.

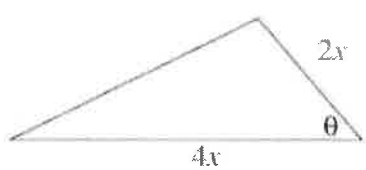


$$(\sqrt{b})^2 = 6^2 + 2^2 - (2)(6)(2)(\cos B)$$

$$b = 36 + 4 - (2)(6)(2)\cdot\frac{3}{8} = 40 - 3 = \boxed{37}$$

Question 16

The area of the triangle shown is 27 sq. units. Given that $\sin\theta = \frac{3}{4}$, find x.



$$\text{Area} = \left(\tfrac{1}{2}\right)(4x)(2x)\cdot\sin\theta$$

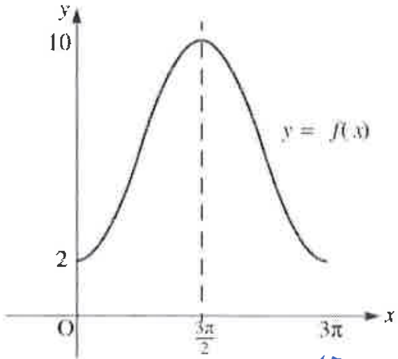
$$27 = 4x^2 \cdot \frac{3}{4}$$

$$27 = 3x^2$$

$$9 = x^2 \Rightarrow x = 3 \quad (x > 0)$$

Question 22

The graph of $f(x) = a\cos(bx) + c$, $0 \le x \le \pi$ is shown below. Find the values a, b and c.



period : 3π

$a = -\left(\frac{\max - \min}{2}\right) = \frac{10-2}{2} = -4$

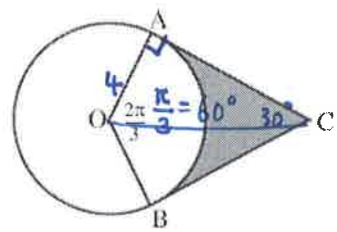
$c = \frac{\max + \min}{2} = \frac{10+2}{2} = 6$

$b = \frac{2\pi}{3\pi} = \frac{2}{3}$

$\Rightarrow y = -4\cos\left(\frac{2}{3}x\right) + 6$

Question 21

The segments [CA] and [CB] are tangents to the circle at the points A and B respectively. If the circle has a radius of 4 cm and $\angle AOB = \frac{2\pi}{3}$.



(a) Find the length of [OC]. $\overline{OC} = 8$ cm

(b) Find the area of the shaded region.

Area of Shaded $= 16\sqrt{3} - \frac{16\pi}{3}$

4, $4\sqrt{3}$, 8

$\Rightarrow \triangle$ Area $= \left(\frac{1}{2}\right)(4)\left(4\sqrt{3}\right) = 8\sqrt{3}$

$\Rightarrow$ Area $= (2)\left(8\sqrt{3}\right) = 16\sqrt{3}$

Area of sector $= \left(\frac{1}{2}\right)(4)^2\left(\frac{2\pi}{3}\right) = \frac{16\pi}{3}$.

Question 28

If $\sin A = \frac{\sqrt{2}}{4}$ and $\cos B = \frac{\sqrt{3}}{4}$ where both A and B are acute angles, find

4, $\sqrt{2}$, $\sqrt{16-2} = \sqrt{12} = 2\sqrt{3}$

4, $\sqrt{3}$, $\sqrt{16-3} = \sqrt{13}$

(a) (i) sin2A.
(ii) cos2B.

(i) $\sin 2A = 2\sin A \cdot \cos A = (2)\left(\frac{\sqrt{2}}{4}\right)\left(\frac{\sqrt{3}}{4}\right) = \frac{\sqrt{6}}{8}$

(ii) $\cos 2B =$

(c) sin(A + B).

(c) $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B = \frac{\sqrt{2}}{4} \cdot \frac{\sqrt{3}}{4} + \frac{2\sqrt{3}}{4} \cdot \frac{\sqrt{13}}{4} = \frac{\sqrt{6} + 2\sqrt{39}}{16}$

Question 30

(a) Let $s = \sin\theta$. Show that the equation $15\sin\theta + \cos^2\theta = 8 + \sin^2\theta$ can be expressed in the form $2s^2 - 15s + 7 = 0$.

(b) Hence solve $15\sin\theta + \cos^2\theta = 8 + \sin^2\theta$ for $\theta \in [0, 2\pi]$.

(a) $15\sin\theta + (1 - \sin^2\theta) = 8 + \sin^2\theta$

$\Rightarrow 2\sin^2\theta - 15\sin\theta + 7 = 0 \Rightarrow 2s^2 - 15s + 7 = 0$. Where $s = \sin\theta$

$2s \quad -1$
$s \quad -7$

$(2s-1)(s-7) = 0 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$s = \frac{1}{2}$ $\quad \cancel{s = 7}$

Question 5

The length, in minutes, of telephone calls at a small office was recorded over a one month period. The results are shown in the table below.

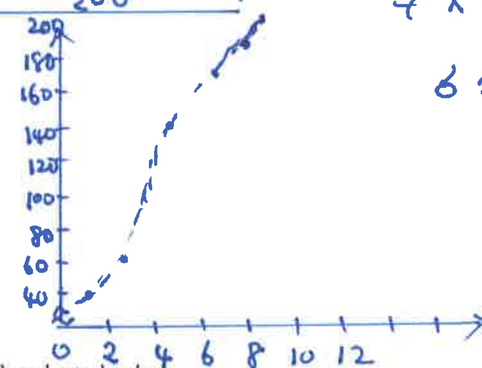
Length of call (minutes)	Number of calls	Cumlative calls
$0 < t \leq 2$	40	40
$2 < t \leq 4$	60	100
$4 < t \leq 6$	40	140
$6 < t \leq 8$	30	170
$8 < t \leq 10$	20	190
$10 < t \leq 12$	10	200
Total	**200**	

mean:

$[(1)(40) + (3)(60) + (5)(40) + (7)(30) + (9)(20) + (11)(10)] \div 200$

$\Rightarrow \bar{x} \approx 4.60$

$\sigma \approx 2.87$



(a) Construct a cumulative frequency graph.

(b) Find (i) the mean length of calls. (i) 4.60
(ii) the mode of the length of calls. (ii) 3.00
(iii) the median. (iii) 5.00

Question 7

A carton contains 12 eggs of which 3 are known to be bad. If 2 eggs are randomly selected, what is the probability that

(a) one is bad?

(b) both are bad?

(a) $\left(\frac{3}{12}\right)\left(\frac{9}{11}\right) + \left(\frac{9}{12}\right)\left(\frac{3}{11}\right) = (\cancel{2})\left(\frac{27}{\cancel{132}\,66}\right) = \frac{27}{66} = \frac{9}{22}$

(b) $\left(\frac{3}{\cancel{12}\,6}\right)\left(\frac{\cancel{2}}{11}\right) = \frac{1}{22}$

Question 15

How many permutations are there of the word RETARD if

(a) they start with RE. $\Rightarrow 4! = 24$

(b) they do not start with RE. $\Rightarrow 6! = 720$

Question 13

Students at Leegong Grammar School are enrolled in either physics or mathematics or both. The probability that a student is enrolled in physics given that they are enrolled in mathematics is $\frac{1}{3}$ while the probability that a student is enrolled in mathematics given that they are enrolled in physics is $\frac{1}{4}$. The probability that a student is enrolled in both mathematics and physics is x.

(a) Find, in terms of x, the probability that a student is enrolled in
  (i) mathematics.
  (ii) physics.

$P(M) = 3x$, $P(P) = 4x$

(b) Find the probability that a student selected at random is enrolled in mathematics only.

$P(M \text{ only}) = \frac{1}{3}$

(c) If three such students are randomly selected, what is the probability that at most one of them is enrolled is mathematics only?

At most one means : one or none.

$\Rightarrow 1 - \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^2 \cdot {}_3C_1 - \left(\frac{1}{2}\right)^3 = \frac{5}{8}$

13.

$\Rightarrow P(P|M) = \frac{1}{3} = \frac{P(P \cap M)}{P(M)}$

$\Rightarrow P(M|P) = \frac{1}{4} = \frac{P(P \cap M)}{P(P)}$

$P(P \cap M) = x$

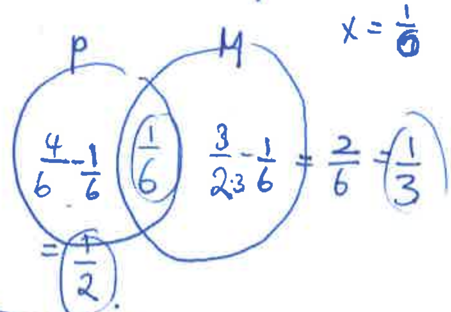
(a) $\frac{1}{3} = \frac{x}{P(M)}$

$\frac{1}{4} = \frac{x}{P(P)}$

$P(M) + P(P) - \underbrace{P(M \cap P)}_{=1}$

$3x + 4x - x = 1$

$x = \frac{1}{6}$



Question 18

A fair coin is tossed five times.

(a) Find the probability of observing
  (i) exactly one tail.
  (ii) at least four tails. 4 OR 5.
  (iii) at least one tail.

5, 4, 3, 2, 1.

18 .

(i) $\left(\frac{1}{2}\right)^5 \cdot {}_5C_1$

(ii) $\left(\frac{1}{2}\right)^5 {}_5C_4 + \left(\frac{1}{2}\right)^5 {}_5C_5$

(iii) $1 - \left(\frac{1}{2}\right)^5$

Question 24

If $P(A) = \frac{3}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cup B) = \frac{7}{8}$, find

(a) $P(A \cap B)$.

(b) $P(A' \cap B)$.

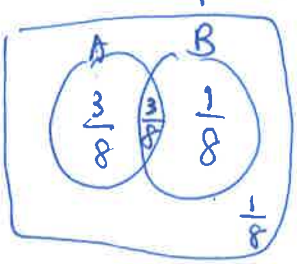
(c) $P(A'|B')$.

$P(A \cup B) = \frac{7}{8} = P(A) + P(B) - P(A \cap B)$

$= \frac{3}{4} + \frac{1}{2} - P(A \cap B)$

$\frac{6}{8} - \frac{4}{8}$

$\Rightarrow P(A \cap B) = -\frac{7}{8} + \frac{3}{4} + \frac{1}{2} = \frac{3}{8}$.



(a) $P(A \cap B) = \boxed{\frac{3}{8}}$

(b) $P(A' \cap B) = \boxed{\frac{1}{8}}$

(c) $P(A'|B') = \frac{P(A' \cap B')}{P(B')} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4}$