**Modeling Constant and Proportional Harvests of a Chum Salmon Population**

Rationale:

When I began looking for a suitable topic for this exploration, due to my deep interest in and affinity for differential equations, I decided to look into how differential equations played a role in my personal life. In my research, I found two differential equations that modeled natural resources as they grew and were harvested at the same time. This sparked my interest because, living in Washington state in the Pacific Northwest, salmon, a natural resource, is an important part of our culture and its sustainability is of utmost importance. This led me to choose this topic and I decided to compare the two different models that I found because, as with any other natural resources, the goal is to sustainably harvest them without causing their extinction or destruction. The two models that I found were the Constant Harvest Model and the Proportional Harvest Model.

The concept behind this exploration is applicable to sustainability practice, which is being used by many environmentally conscious companies, and has far-reaching implications in the ways fishing companies structure their harvest patterns, grounding this exploration in a realistic situation. By modeling each model using a differential equation, I will evaluate the impacts of using each type of harvest and determine the more sustainable model by a comparison of their slope fields in addition to a prediction of the chum salmon population for the next 5 years. While the prediction shows the immediate change in the population, the slope fields indicate the stability of each solution, allowing me to determine the model that is least likely to lead to extinction over a longer period of time. With that in mind, the aim of this exploration is to determine which of the Constant Harvest and Proportional Harvest models is the most sustainable over time.

Introduction:

Even though fish is a regular part of human life and the human diet for many, what exactly do we know about the process of fishing? What factors affect the growth and decay of a fish population that is being harvested? There are multiple constant factors associated with the populations of fish:

1. Carrying capacity: This is the highest number of fish that the specific ecosystem that they inhabit can hold
2. Growth rate: This is the rate at which the fish reproduce and according to the law of exponential growth, this is a proportionality constant to the size of the fish population (i.e. the population of fish grows at a rate proportional to the size of the population)
3. Harvest rate: This is the rate at which the fish are removed from the population. For the purpose of this exploration, this is expressed in two ways (Boyce and Prima, 90):
	1. In the Constant Harvest Model, it is simply expressed as a constant, unaffected by the size of the population at any time
	2. In the proportional harvest model, it is expressed as a proportionality constant similar to the growth rate (i.e. the rate at which the fish are harvested is proportional to the size of the population)

These equations are further elaborated upon in the next section.

For the sake of simplicity, these factors are assumed to be constants in the models. Furthermore, as an estimate of how animal populations grow, the growth of the fish population is modeled using Verhulst’s Logistic Growth Equation.

First Model: Constant Harvest Model

This model is written in the form of a differential equation as:

In this equation:

* is the overall rate of change of the population with respect to time
* is the size of the fish population
* is the time in years after the initial time
* is the intrinsic growth proportionality constant
* is the constant carrying capacity of the environment
* is the constant harvest rate, independent of any variable

The implication of using the logistic growth model in the equation is that the population cannot grow indefinitely unlike the law of exponential growth. To ground this exploration in reality, the logistic model makes it so that the population approaches, but does not extend beyond the carrying capacity (see Fig 1.1.). In the equation, we can see that as , . As the population approaches the carrying capacity, the growth rate of the population decreases (“Exponential & Logistic Growth”).

To model the population of the fish, I first need to solve this differential equation. However, this equation is not easily differentiable and for the sake of simplicity, I have decided to approximate a solution to this differential equation using Euler’s Method of Approximation. However, I first need to determine the value of the constants in the equations. To make this model more realistic, I will use data from the most recent year collected in a report by professors at Hokkaido University, Japan, which is 1993. According to the data, for the year of 1993 (Kaeriyama et al.):

Fig 1.1. shows a sample logistic curve where the population approaches, but does not exceed the carrying capacity

* The population of wild chum salmon was 71.9 million
* The annual growth rate (*g*) of the chum salmon population was 50% or 0.5
* The carrying capacity () of the wild chum salmon population, without accounting for the hatchery population, was 117.2 million. For the sake of simplicity in modeling, I will simply use 117.2 as the *k* value rather than 117.2 million and the population values will also be reported in millions at the same degree of accuracy to the nearest 0.1 million.

Since the goal of this exploration is to determine the harvest rate at which the chum salmon population will be stable and sustainable, I will find the equilibria solutions to this differential equation where . When this occurs, it means that the rate at which the population grows is equal to the rate at which the population is harvested, and this will occur at the equilibria population solutions. Therefore:

, rewriting this as a quadratic equation and multiplying by -1,

, multiplying the equation by ,

, then applying the quadratic formula, we see that the equilibria solutions to this differential equation are:

From this solution, we can see that:

* If , there are two real solutions for the equilibria population
* If , there is one real solution for the equilibria population
* If , there are no real solutions for the equilibria population

Rewriting this with the constants, the equilibria solutions are:

: I decided to graph these solutions in a bifurcation diagram to visualize the possible *h* values.

Graph 1: Bifurcation Diagram of *P* and *h*



*P*

Bifurcation Diagram of *P* and *h*

*h*

Equilibria Population Solutions – (millions of fish)

Constant Harvest Rate – (millions of fish)

The x-coordinate where the two solutions meet is the bifurcation point for which there is only equilibrium solution. At this point, , as it has only real solution at . Any x-coordinate greater than this results in no real solution and any x-coordinate less than this has 2 equilibria solutions, one greater than and one less than . Since the bifurcation point has the highest possible equilibrium solution that does not have a second, lower equilibrium solution, it provides the most sustainable chum salmon population of all the possible constant harvest rates. Therefore, the bifurcation point will be used to determine *h*. Using the bifurcation point, Now, the differential equation is fully written:

, where *P* is in millions of chum salmon.

To predict the population after 5 years, I will use Euler’s Method of Approximation, which states:

, where is the initial population (71.9 million), is the population after 1 year, is the change in time or the step (1 year), and when and .

For example, if the initial population is 71.9 million in 1993, the population after 1 year can be written as:

 in millions of chum salmon

Table 1: Euler’s Approximation of the Constant Harvest Model for 5 Years

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **n (Step number)** | 0 | 1 | 2 | 3 | 4 | 5 |
|  **(years)** | 0 | 1 | 2 | 3 | 4 | 5 |
|  **(in millions of fish)** | 71.9 | 71.1 | 70.5 | 69.9 | 69.3 | 68.8 |

This table shows that the population exhibits an immediate decline towards the one real equilibrium solution of 58.6 million fish. However, this is not enough to determine whether the Constant Harvest Model is sustainable. To truly determine if this is sustainable, the stability of the equilibrium solution must be verified by plotting the equilibrium solution and the slope field of the differential equation:

Graph 2: Slope Field of the Constant Harvest Model



d*P*/d*t*

Slope Field of the Constant Harvest Model

Rate of Population Change

Time – (number of years after 1993)

*t*

This slope field shows that while 58.6 million is a solution to the equilibrium population, it is neither a sink nor a source solution, but it is a node. When the population is higher than the equilibrium 58.6 million, the population will gravitate towards the equilibrium population. However, if the initial population started at a lower value, the constant harvest rate would always lead to the extinction of the chum salmon population, indicated by the negative slopes when *P* < 58.6 in millions of fish.

Furthermore, the node solution poses another threat to the chum salmon population. If the population declines from its current value to the equilibrium population, there is a plausible chance that a natural disaster or a different real world factor can cause the chum salmon population to fall beneath the node solution of 58.6 million. If this happens, the population will inevitably result in extinction and the only feasible way to combat this would be to change the constant harvest rate. However, changing the constant harvest rate undermines the goal of having a harvest rate that does not need to be changed to maintain a sustainable population. Therefore, the Constant Harvest Model is not viable as a sustainable model for the chum salmon population.

Second Model: Proportional Harvest Model

This model follows the same principle as the Constant Harvest Model, with the sole difference being, the harvest rate is also a proportionality constant to the population just as the growth rate:

In this equation, all variables remain the same other than *E*, which represents the proportionality harvest constant. Just as in the Constant Harvest Model, the equilibria solutions must be determined:

, distributing *gP* and rewriting as a quadratic equation,

, multiplying the equation by ,

, combining the coefficients of *P* and factoring out *k*,

, then applying the quadratic formula, we see that the equilibria solutions to this model are:

Solution 1:

Solution 2:

From Solution 2, we can see that:

* If , there is one positive, non-zero equilibrium solution
* If , there is only one equilibrium solution at *P* = 0
* If , there is one negative, non-zero equilibrium solution

Since population cannot be negative, the solution of will be discarded. Furthermore, since the goal of this exploration is to maintain a sustainable non-zero population, the solution of will also be discarded. Therefore to maintain a sustainable non-zero population.

To maintain consistency with the analysis of the Constant Harvest Model, I will determine a value for *E*, letting the equilibrium population solution be :

, dividing both sides by *k*,

, subtracting from and adding to both sides of the equation,

, substituting for the known constant ,

This also agrees with the earlier assertion that .

Before making use of this, we must determine a method to predict the population of chum salmon over 5 years. For the Constant Harvest Model, Euler’s Method of Approximation was used because the differential equation was not easily differentiable. However, the Proportional Harvest Model is separable and differentiable:

, rewriting as and factoring out ,

, factoring from ,

, substituting and to simplify the process,

, dividing both sides by and multiplying both sides by ,

, using partial fractions to split the left term,

, multiplying both sides by *M* and integrating both sides,

, where *C* is the constant of integration; multiplying both sides by -1,

, using the laws of natural exponents to combine the two left terms,

, raising both sides as powers of *e*,

, letting since it is a constant,

, separating the left side of the equation into two terms,

, adding 1 to both sides of the equation,

, multiplying both sides by *P* and dividing both sides by ,

, substituting back for and ,

Now, the known constants will be substituted back into the equation to determine a definitive model:

To determine *B*, the initial *t* (*t* = 0) and *P* (*P* = 71.9, in millions of fish) values will be substituted into the equation:

, simplifying the denominator of the right term,

, multiplying both sides by *B* + 1 and dividing both sides by 71.9,

, subtracting 1 from both sides,

Therefore, the definitive population model for the Proportional Harvest Model is:

This will be used to determine the population over the 5 successive years from 1993 by substituting values 1-5 for *t*.

Table 2: Population Values for the Proportional Harvest Model for 5 Years

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| ***t* (years)** | 0 | 1 | 2 | 3 | 4 | 5 |
| ***P* (in millions of fish)** | 71.9 | 68.5 | 66.0 | 64.2 | 62.9 | 61.9 |

From this table of values, I was quite surprised to see that the decline in the population was far higher than in the Constant Harvest Model because this model is designed to change the harvest rate depending on the population, which I thought would imply that the decline would be more gradual than the Constant Harvest Model. However, keeping in mind that the goal of this exploration is to determine the sustainability of this solution over time, I decided to plot the slope field of this differential equation to verify its stability.

Graph 3: Slope Field of the Proportional Harvest Model

d*P*/d*t*

Slope Field of the Proportional Harvest Model



Rate of Population Change

*t*

Time – (number of years after 1993)

This slope field shows that, unlike the Constant Harvest Model, the equilibrium solution of 58.6 million is not a node or a source, but a sink. This implies that if the non-zero population of chum salmon is either greater or lesser than the equilibrium population, it will gravitate towards the equilibrium solution as time passes.

Unlike the node solution to the Constant Harvest Model, if a natural disaster reduces the chum salmon population to a number lower than the equilibrium population, the Proportional Harvest Model will allow it to recover back to the equilibrium population rather than lead it to extinction, indicated by the positive slopes when *P* < 58.6 in millions of fish. This implies that even though the initial decrease in the chum salmon population is greater for the Proportional Harvest Model, over time, it is still the most sustainable population model for the chum salmon population.

Reflection:

While I could definitively determine which of the two models was more sustainable, there remain a few discrepancies and limitations with my modeling. First, it does not consider the fact that the carrying capacity of an environment changes over time due to causes such as the expansion of industry or natural disasters. This forced me to choose a singular, constant value for the carrying capacity, making these models quite unrealistic. Furthermore, it also does not consider any real-life factors that would cause fluctuations in this population such as natural disasters. An interesting area for further investigation would be to incorporate a minimum threshold value and a changing carrying capacity. If the population were to drop to a number lower than this minimum threshold value, the population would lead to extinction. Perhaps two graphs could be plotted modeling the impact of a changing carrying capacity and the minimum threshold value on the equilibrium population solutions for a fixed growth and harvest rate. However, investigating this would involve comparisons of bi-variant data, which would render problem-solving and modeling far more difficult.

Despite these minor discrepancies that take away from the applicability of these models in real-life situations, I have still achieved the goal of my exploration, which was to determine which of the Constant Harvest and the Proportional Harvest models is the most sustainable over time. The Proportional Harvest Model, although exhibiting a sharper initial decline, is shown to be the more sustainable of the two due to its ability to aid a population to recover if an undefined factor causes the population to drop to a number lower than the equilibrium solution value.

Works Cited:

* Boyce, William E., and Richard C. Prima. Elementary Differential Equations and Boundary Value Problems. 8th ed., Hoboken, John Wiley & Sons, 2005.
* "Exponential & Logistic Growth." Khan Academy, 2017, www.khanacademy.org/science/biology/ecology/population-growth-and-regulation/a/exponential-logistic-growth. Accessed 14 May 2017.
* Kaeriyama, Masahide, et al. "Trends in Run Size and Carrying Capacity of Pacific Salmon in the North Pacific Ocean." ResearchGate, researchgate.net, Jan. 2009, www.researchgate.net/publication/268432164\_Trends\_in\_Run\_Size\_and\_Carrying\_Capacity\_of\_Pacific\_Salmon\_in\_the\_North\_Pacific\_Ocean. Accessed 14 May 2017.

Graphing Software:

* fooplot.com (used for Bifurcation Diagrams)
* GeoGebra (used for Slope Fields)