IB Pre HL    6E: Sum and Product of Roots Theorem    Name: ____________________

Exploration:

1) Given $a_2(x-\alpha_1)(x-\alpha_2)=0$,

$a_2[x^2+x(-\alpha_1-\alpha_2)+\alpha_1\cdot\alpha_2]$

a) Write the equation in the form of $x^2+Sx+P=0$.

$\frac{a_2x^2}{a_2}+\frac{a_2x(-\alpha_1-\alpha_2)}{a_2}+\frac{a_2(\alpha_1\cdot\alpha_2)]}{a_2}=\frac{0}{a_2}$

Answer: ____________________

b) What does S equal to? $-(\alpha_1+\alpha_2)$ ? What does P equal to? $\alpha_1\cdot\alpha_2$

2) Given $a_3(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)=0$,

a) Write the equation in the form of $x^3+Sx^2+Wx+P=0$.

Answer: ____________________

b) What does S equal to? $-(\alpha_1+\alpha_2+\alpha_3)$ ? What does P equal to? $-\alpha_1\alpha_2\alpha_3$

3) Given $a_4(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)=0$,

a) Write the equation in the form of $x^4+Sx^3+Wx^2+Zx+P=0$.

Answer: ____________________

b) What does S equal to? $-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$ ? What does P equal to? $\alpha_1\alpha_2\alpha_3\alpha_4$

4) Given $a_5(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)=0$,

a) Write the equation in the form of $x^5+Sx^4+Wx^3+Zx^2+Qx+P=0$.

Answer: ____________________

b) What does S equal to? $-(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)$ What does P equal to ? $-\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$

5) Observing the patterns of above exercise, what do S and P to, for the 5the degrees of polynomial of $x^6+Sx^5+Wx^4+Zx^3+Qx^2+Rx+P=0$ if it is factored to be

$a_5(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)(x-\alpha_6)=0$

S= $-(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)$    P= $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$

For the polynomial equation, $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \ldots\ldots + a_0 = 0$

(which can also be written as $\sum_{r=0}^{n} a_r x^r = 0$) and where $a_n \neq 0$

- The sum of the roots is $\left( -\dfrac{a_{n-1}}{a_n} \right)$.
- The product of the roots is $\left( -\dfrac{a_0}{a_n} \right)$ if n is odd.
- The product of the roots is $\left( \dfrac{a_0}{a_n} \right)$ if n is even.

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- For the polynomial equation, $a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0$
- $\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2.\alpha_3 = \dfrac{a_1}{a_3}$ where $\alpha_1, \alpha_2, and\, \alpha_3$ are the roots of the polynomial

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Example 1) Find the sum and product of the roots of $2x^3 - 7x^2 + 8x - 1 = 0$

$\text{Sum} = \dfrac{-(-7)}{2} = \boxed{\dfrac{7}{2}}$ $\qquad$ $\text{product} = -\left(\dfrac{-1}{2}\right) = \boxed{\dfrac{1}{2}}$

Example 2) A real polynomial has the form $P(x) = 3x^4 - 12x^3 + cx^2 + dx + e$. The graph of $y = P(x)$ has y-intercept 180. It cuts the x-axis at 2 and 6, and does not meet the x-axis anywhere else. Suppose the other two zeros are $m \pm ni$, $n > 0$. Use the sum and product formulae to find m and n.

$e = 180 \qquad x_1 = 2, \quad x_2 = 6$

$x_3 = m + ni \qquad x_4 = m - ni$

$\text{Sum}: \; 2 + 6 + (m + \cancel{ni}) + (m - \cancel{ni}) = \dfrac{12}{3} = 4$

$8 + 2m = 4 \qquad 2m = -4 \qquad m = -2$

$\text{product}: \; (2)(6)(-2 + ni)(-2 - ni) = \dfrac{180}{3} = 60$

$\cancel{48}\,12\,[4 - (ni)^2]$

$12(4 + n^2) = 60$

$4 + n^2 = 5$

$n^2 = 1 \qquad n = \cancel{\pm}1$

$n = 1$

$m = -2$

Example 3)

Given that the roots of a cubic equation $2x^3 + 4x^2 - 7x + 5 = 0$ are $x_1, x_2$ and $x_3$, without solving the equation, find:

**a** $x_1 + x_2 + x_3$  **b** $x_1 \cdot x_2 \cdot x_3$  **c** $x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3$

**d** $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$  **e** $x_1^2 + x_2^2 + x_3^2$

a) $x_1 + x_2 + x_3 = \frac{-4}{2} = -2$.

b) $x_1 x_2 \cdot x_3 = \frac{-5}{2}$

c) $x_1 x_2 + x_1 x_3 + x_2 x_1 = \frac{-7}{2}$.

d) $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{x_2 x_3 + x_1 x_3 + x_1 x_2}{x_1 \cdot x_2 x_3} = \frac{\frac{-7}{2}}{\frac{-5}{2}} = \frac{+7 \cdot \cancel{2}}{\cancel{2} \cdot +5} = \boxed{\frac{7}{5}}$

e) $x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1 x_2 + x_2 x_3 + x_1 x_3)$

$= (-2)^2 - 2\left(\frac{-7}{2}\right) = \boxed{11}$

More Practice) Work in your notes

1. See attached

The roots of a quadratic equation $2x^2 + 4x - 1 = 0$ are $\alpha$ and $\beta$.

Without solving the equation,

(a) find the value of $\alpha^2 + \beta^2$;

(b) find a quadratic equation with roots $\alpha^2$ and $\beta^2$.

2.

Let $p(x) = 2x^5 + x^4 - 26x^3 - 13x^2 + 72x + 36$, $x \in \mathbb{R}$.

(a) For the polynomial equation $p(x) = 0$, state

(i) the sum of the roots;

(ii) the product of the roots.

A new polynomial is defined by $q(x) = p(x + 4)$.

(b) Find the sum of the roots of the equation $q(x) = 0$.

#1 More practice answers.

a) $\alpha + \beta = \frac{-4}{2} = -2$

$\alpha\beta = \frac{-1}{2}$

$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$.

$\Rightarrow \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (-2)^2 - (2)(\frac{-1}{2}) = 4 + 1 = \boxed{5}$

b) Sum: $\alpha^2 + \beta^2 = 5$

product: $\alpha^2 \cdot \beta^2 = (\alpha \cdot \beta)^2 = (\frac{-1}{2})^2 = \frac{1}{4}$

$\Rightarrow x^2 - 5x + \frac{1}{4} = 0 \Rightarrow 4x^2 - 20x + 1 = 0$

#2. $p(x) = 2x^5 + x^4 - 26x^3 - 13x^2 + 72x + 36$

$n = 5$

1 0
1 1 1
1 2 1 2
1 3 3 1 3
1 4 6 4 1 4
$\boxed{1\ 5\ 10\ 10\ 5\ 1}$ 5

$\rightarrow$ (a) (i) Sum: $\boxed{-\frac{1}{2}}$

(ii) product: $\frac{-36}{2} = \boxed{-18}$

(b) $q(x) = 2(x+4)^5 + (x+4)^4 - 26(x+4)^3 - 13(x+4)^2 + 72(x+4) + 36$

$= 2(x^5 + \boxed{5x^4(4)} + 10x^3(4)^2 \cdots) + (\boxed{x^4} + 4x^3 \cdots)$

Sum: $\boxed{\frac{-41}{2}}$

$\frac{2 \cdot 5 \cdot 4x^4 + x^4}{2} = -\frac{41}{2}[x^4]$